Hint To The Exercises

1. We assume that $f:\mathbb{C}^n \to \mathbb{C}$ is *RWH* of type $(q_1,\dots,q_n;d_r)$. Take $t\in\mathbb{R}^*$ and consider the radial action as follows
$$f(t^{q_1}z_1,\dots,t^{q_n}z_n,t^{q_1}\overline{z_1},\dots,,t^{q_n}\overline{z_1}) = t^{d_r}f(\mathbf{z},\overline{\mathbf{z}}).$$
Now, take the derivatives with respect to t of both sides in above equality. Then the identity follows by putting $t=1$. For *PWH* mixed polynomials, the proof is similar.
2. Given a mixed polynomial germ $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$, we may consider $f = g+\sqrt{-1}h$ as a real polynomial mapping germ (g,h) of $2n$-variables $(x_1,y_1,\dots,x_n,y_n)$, where $z_j = x_j+\sqrt{-1}y_j$. Then the Milnor set of f is exactly the same as the Milnor set of (g,h). Therefore, it remains to rewrite the equation of $M(g,h)$ by using the coordinates of $\mathbf{z}$ and $\overline{\mathbf{z}}$.
3. In Oka's example, the mixed polynomial f is *RWH* of type $(1,1;2)$. By using the formula of mixed singularity(cf. Proposition 2.2 of Lecture 1), we have the singular locus of f as follows,
$$\operatorname{Sing} f = \{z_1=0\}\cup\{z_1=\pm i\bar{z}_1, z_1+z_2=0\}.$$
Let us choose the weight vector $P=(1,1)$, then the face function associated with P is just f. Since $\operatorname{Sing} f\cap\mathbb{C}^{*2}\neq\emptyset$, the polynomial f is not strongly non-degenerate. In fact, there are three faces on the Newton boundary of f which correspond to the weight vectors $E_1=(1,0)$, $P=(1,1)$ and $E_2=(0,1)$. By using the formula of mixed singularity, we can easily prove the non-degeneracy condition for f.
4. Let us consider the following mixed polynomial $f:\mathbb{C}\to\mathbb{C}$,
$$f(z,\bar{z}) = az^2+bz\bar{z}+c\bar{z}^2,$$
where $a,b,c\in\mathbb{C}$. Then f is strongly non-degenerate if and only if the following inequality holds
$$(|a|^2-|c|^2)^2 > |\bar{a}b-c\bar{b}|^2.$$
Hence this example shows the strong non-degeneracy condition for mixed polynomial is neither dense nor connected. However, the strong non-degeneracy condition is a semi-algebraic open condition with respect to the fixed Newton Boundary.
5. The answer is yes! For a polar weighted homogenous mixed polynomial $f:\mathbb{C}^n\to\mathbb{C}$, we can show that f is locally a surjective. In fact, we may assume there is a curve $\gamma\in\operatorname{Im}(f)$ which passes through the origin.(Hint: Curve selection lemma) By using the polar action, if $c\in\operatorname{Im}(f)$, then $\lambda c\in\operatorname{Im}(f)$ for any $\lambda\in S^1$. Therefore, it shows that locally $\operatorname{Im}(f)$ contains a small disk D_δ centered at $0\in\mathbb{C}$. The same reasoning shows that if $a\in\mathbb{C}$ is a regular value of f(resp. $f_{|S_\varepsilon^{n-1}}$), then λa is a regular value for any $\lambda\in S^1$. Therefore f satisfies Milnor's Condition A and B, which implies the existence of Milnor tube fibration.
6. At first, we consider the following regular simplicial fan for the dual Newton diagram,
$$\Sigma = \left\{E_1=\begin{bmatrix}1\\0\end{bmatrix}, P_1=\begin{bmatrix}1\\1\end{bmatrix}, P_2=\begin{bmatrix}2\\3\end{bmatrix}, P_3=\begin{bmatrix}1\\2\end{bmatrix}, E_2=\begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

For instance, we take the cone $\sigma = \mathrm{Cone}(P_2, P_3)$. The strict transformation of the toric modification associated with σ is

$$\tilde{V} = \{(u_1, u_2) \in \mathbb{C}_\sigma^{*2} | \bar{u}_2 \bar{u}_1^2 - 2u_1^2 u_2^2 = 0\}$$

where (u_1, u_2) is the toric coordinate in $\mathbb{C}_\sigma^2$. For the next, we use the polar blowing up to resolve the singularity. We take the polar coordinate $(r_1, \theta_1, r_2, \theta_2)$. Then the strict transformation of the polar blowing up is

$$\hat{V} = \{(r_1, \theta_1, r_2, \theta_2) | 2r_2 \exp(3i\theta_2) - \exp(-4i\theta_1) = 0\}.$$

Therefore $r_2 = \frac{1}{2}$, and the exceptional divisor is $\hat{E}(\sigma) = \{r_1 = 0\}$. For the resolution of singularity in the other coordinate charts, the procedure is similar. We leave the verification to the reader.

7. Note that f is a good polar weighted homogenous mixed polynomial of polar type $(2, 1; 4)$. Moreover, f is convenient and strongly non-degenerate. Hence the number of link components $\mathrm{lkn}(f^{-1}(0), 0)$ is coincide with $\mathrm{lkn}^*(f^{-1}(0), 0)$. By using the formula(cf. Lecture 2, pp.16 and pp.20), we have $\mathrm{lkn}^*(f^{-1}(0), 0) = 2$ and the Milnor number $\mu(f) = \mathrm{lkn}^*(f^{-1}(0), 0) + 1 = 3$. Therefore by Theorem E, the zeta function of the monodromy map is
$$\zeta(t) = (1 - t^4)^2(1 + t^2).$$
8. To give a non-trivial example in the mixed case, we proposed the following mixed Brieskorn's polynomial
$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_1^{a_2} + z_2^{b_1} \bar{z}_2^{b_2}$$