Milnor Fibration Of Mixed Polynomials(I)

The Bridge Between Real And Complex Singularity Theory

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- Every non constant complex polynomial $f(z_1, \ldots, z_m)$ with an isolated singularity at the origin gives $f : (\mathbb{R}^{2m}, 0) \to (\mathbb{R}^2, 0)$ which satisfies the Hypothesis*.

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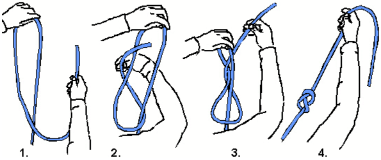
1. Every non constant complex polynomial $f(z_1, \ldots, z_m)$ with an isolated singularity at the origin gives $f : (\mathbb{R}^{2m}, 0) \to (\mathbb{R}^2, 0)$ which satisfies the Hypothesis*.
2. Hypothesis* is a sufficient condition to guarantee the existence of Milnor tube fibration. We often call Hypothesis* as,

$$f \text{ has an isolated singularity at } x = 0.$$

Problem

Are there other, essentially different, examples for $k = 2$?



An Instructive Example

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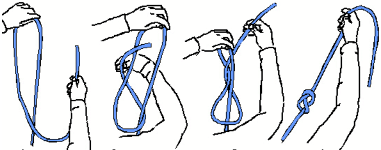
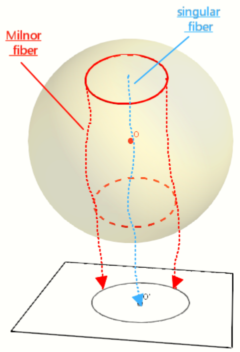


[Rudolph's example, 1987]: Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a complex valued function, $F(z, \omega) = \omega^3 - 3\|z\|^2(1 + z - \bar{z})\omega - 2(z + \bar{z})$.
Consider $G(z, \omega) = F(z^2, \omega)$ as a real polynomial mapping germ from $\mathbb{R}^4$ to $\mathbb{R}^2$. Then G has an isolated singularity at $(0, 0)$. Denote the link of singularity at $(0, 0)$ by $K(G)$. Quite strikingly,

$K(G)$ is the figure-8 knot.

What's About?

This mini-course is about



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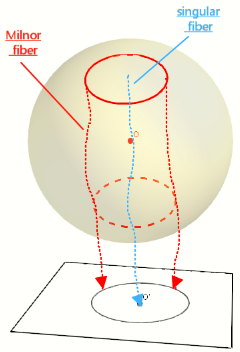

Milnor fibration of mixed polynomials.

Mixed Polynomials

A *mixed polynomial* $f : \mathbb{C}^n \to \mathbb{C}$ is written as follows,

$$f(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\upsilon,\mu} c_{\upsilon,\mu} \mathbf{z}^{\upsilon} \overline{\mathbf{z}}^{\mu}$$

where $\mathbf{z} = (z_1, \ldots, z_n)$, $\overline{\mathbf{z}} = (\overline{z_1}, \ldots, \overline{z_n})$, and $\mathbf{z}^{\upsilon} = z_1^{\upsilon_1} \cdots z_n^{\upsilon_n}$ for $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$ (respectively $\overline{\mathbf{z}}^{\mu} = \overline{z_1}^{\mu_1} \cdots \overline{z_n}^{\mu_n}$).
Therefore if we write $z_i = x_i + \sqrt{-1} y_i$ and $f = g + \sqrt{-1} h$, then the mixed polynomial f can be regarded as a real polynomial map (g, h) of $2n$-variables $x_1, y_1, \ldots, x_n, y_n$.

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Remark 1.2

The terminology mixed polynomial was firstly used by M.Oka.
If we restrict ourselves to the local case and consider the convergent power series $f(\mathbf{z}, \overline{\mathbf{z}})$ of $\mathbf{z}$ and $\overline{\mathbf{z}}$, then we also call f a *mixed function*.

Homogenous Mixed Polynomials

Given $f(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\upsilon,\mu} c_{\upsilon,\mu} \mathbf{z}^{\upsilon} \overline{\mathbf{z}}^{\mu}$, there are two types of homogeneity

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- $\exists$ integers $q_1, \ldots, q_n \geq 0$(radial weights), $d_r > 0$(radial degree):

$$f(t^{q_1}z_1, \ldots, t^{q_n}z_n, t^{q_1}\overline{z_1}, \ldots, , t^{q_n}\overline{z_1}) = t^{d_r}f(\mathbf{z}, \overline{\mathbf{z}})$$

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- $\exists$ integers $p_1, \ldots, p_n \geq 0$(polar weights), $d_p > 0$(polar degree)

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  for any $\lambda \in \mathbb{C}^*$ with $\|\lambda\| = 1$. We call f a polar weighted homogenous polynomial of type $(p_1, \ldots, p_n; d_p)$.

Proposition 2.1 (Euler Identities)

Let $f(\mathbf{z}, \overline{\mathbf{z}})$ be a mixed polynomial. Then:

(i) *If f is RWH of type $(q_1, \ldots, q_n; d_r)$,*

$$d_r f(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{i=1}^{n} \left(q_i z_i \frac{\partial f}{\partial z_i}(\mathbf{z}, \overline{\mathbf{z}}) + q_i \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(\mathbf{z}, \overline{\mathbf{z}})\right);$$

(ii) *If f is PWH of type $(p_1, \ldots, p_n; d_p)$,*

$$d_p f(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{i=1}^{n} \left(p_i z_i \frac{\partial f}{\partial z_i}(\mathbf{z}, \overline{\mathbf{z}}) - p_i \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(\mathbf{z}, \overline{\mathbf{z}})\right).$$

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- Proof $\subset$ Exercises!

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Note that $\frac{\partial g}{\partial z_i} = \frac{1}{2}(\frac{\partial g}{\partial x_i} - \sqrt{-1}\frac{\partial g}{\partial y_i})$ and $\frac{\partial g}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial g}{\partial x_i} + \sqrt{-1}\frac{\partial g}{\partial y_i})$. Therefore Jacobian $J(g, h)$ can be written by variables $\mathbf{z}$ and $\bar{\mathbf{z}}$.

Proposition 2.2

For a mixed polynomial $f : \mathbb{C}^n \to \mathbb{C}$, $\omega \in \operatorname{Sing} f$ *if and only if there exists* $\lambda \in \mathbb{C}$ *with* $\|\lambda\| = 1$ *such that*

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Example 1

Let $f = z_1^2 \bar{z}_1 + z_2^2 \bar{z}_2$. Then f is not only *RWH* of type $(1, 1; 3)$, but also *PWH* of type $(1, 1; 1)$. Moreover, $\operatorname{Sing} f = \{(0, 0)\}$.

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- A mixed singularity is not necessary to be the singularity of the variety. For example, take $f = \|z_1\|^2 + \|z_2\|^2$. Then $\mathrm{Sing}\, f = \mathbb{C}^2$, whereas the sphere $f = 1$ has no singularities.

Newton Polygon

Assume $f(\mathbf{z},\overline{\mathbf{z}}) = \sum_{\upsilon,\mu} c_{\upsilon,\mu}\mathbf{z}^{\upsilon}\overline{\mathbf{z}}^{\mu}$ with $c_{0,0} = 0$ so that $\mathbf{0} \in f^{-1}(0)$. The Newton Polygon $\Gamma_+(f)$ is defined to be the convex hull of

$$\bigcup_{c_{\upsilon,\mu}\neq 0} (\upsilon + \mu) + \mathbb{R}^{+n},$$

where $\mathbb{R}^{+n} = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_i \geqslant 0,\ i = 1,\ldots,n\}$.
The Newton Boundary $\Gamma(f)$ is defined to be the union of compact faces of $\Gamma_+(f)$. For a positive integer vector $P = (p_1,\ldots,p_n)$, we associate the linear function

$$\ell_P(\upsilon) = \sum_{i=1}^{n} p_i \upsilon_i, \qquad \upsilon \in \Gamma(f).$$

We denote by $P \in \mathbb{N}^+$(resp. $P \in \mathbb{N}^{++}$) for a positive(resp. strictly positive) integer vector.

Face Function

Given $P \in \mathbb{N}^+$, there is a unique face $\Delta(P) \subset \Gamma(f)$ such that ℓ_P takes its minimal value $d(P)$. The face function $f_P(\mathbf{z}, \overline{\mathbf{z}})$ is

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Let $f = z_1^2 \bar{z}_1 + z_1 z_2 + z_2^2 \bar{z}_2$. The two faces Δ_1 and Δ_2 coincide with the vectors $P_1 = (2, 1)$ and $P_2 = (1, 2)$.

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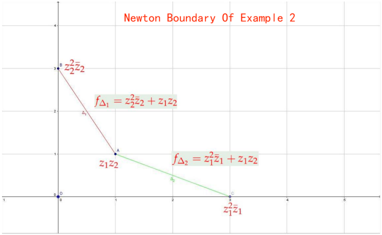

Non-Degeneracy

A mixed polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is *non-degenerate for $P \in \mathbb{N}^{++}$*, if $f_P^{-1}(0) \cap \operatorname{Sing} f_P \cap \mathbb{C}^{*n} = \emptyset$. In particular, if $\operatorname{Sing} f_P \cap \mathbb{C}^{*n} = \emptyset$, we say f is *strongly non-degenerate for $P \in \mathbb{N}^{++}$*. A *(strongly) non-degenerate* mixed polynomial is *(strongly) non degenerate* for all $P \in \mathbb{N}^{++}$.

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[Oka's example, 2009]: Consider the following

$$f = \frac{1}{4}(z_1^2 - \bar{z}_1^2) + \|z_1\|^2 - (1+i)\|z_1 + z_2\|^2.$$

Then f is non-degenerate but not strongly non-degenerate.

For $I \subset \{1, \ldots, n\}$, we write $\mathbb{C}^I = \{z_j = 0,\ j \notin I\}$ and $f^I = f_{|\mathbb{C}^I}$.

Proposition 3.1

Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a (strongly) non-degenerate mixed polynomial. For $I \subset \{1, \ldots, n\}$, if $f^I \not\equiv 0$, then f^I is a (strongly) non-degenerate mixed polynomial with respect to $\{z_i | i \in I\}$.

More About Non-Degeneracy

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Consider the set of non vanishing coordinate subspaces of f

$$NV(f) = \{I \subset \{1, \ldots, n\} | f^I \not\equiv 0\}.$$

We say f is k-convenient if $I \in NV(f)$ for any $I \subset \{1, \ldots, n\}$ with $|I| = n - k$. When $k = n - 1$, we say f is convenient, which is equivalent to that $\Gamma(f)$ intersects with all the coordinate axes.

Theorem A

Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a strongly non-degenerate convenient mixed polynomial. Then f has Milnor tube fibration.

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Remark 4.1

In above Lemma, we can not replace strong non-degeneracy by non-degeneracy. For instance, in Oka's example, the polynomial f is non-degenerate and convenient, whereas it has non isolated singularity at the origin.

Proof Of Lemma A (I)

To obtain the contradiction, by using Curve Selection Lemma, we assume there is an analytic path $\mathbf{z}(t) \subset \operatorname{Sing} f$ with $\mathbf{z}(0) = \mathbf{0}$ for $t \in [0,1]$. Then there are following two cases.

(a) If $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \not\equiv 0$, then by Proposition 2.2 we can find an analytic path $\lambda(t) \in \mathbb{C}$ with $|\lambda(t)| = 1$ such that

$$\overline{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))} = \lambda(t)\bar{d}f(\mathbf{z}(t), \bar{\mathbf{z}}(t)). \tag{4.1}$$

Take $I = \{i | z_i(t) \not\equiv 0\}$. There is no loss of generality in assuming $I = \{1, \ldots, m\}$. Since $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = f^I(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \not\equiv 0$, it follows from Proposition 3.1 that f^I is strongly non-degenerate. Consider the following expansions for $i \in I$

$$z_i(t) = b_i t^{p_i} + (\text{Higher terms}), \qquad b_i \neq 0 \tag{4.2}$$

$$\lambda(t) = \lambda_0 + \lambda_1 t + (\text{Higher terms}), \qquad \|\lambda_0\| = 1. \tag{4.3}$$

Proof Of Lemma A (*II*)

Put $P = (p_1, \ldots, p_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$. Suppose $f^I_{\Delta_P}$ is the face function for $\Delta_P \subset \Gamma(f^I)$ with $d = d(P)$. Then for $i \in I$

$$\frac{\partial f}{\partial z_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f^I_{\Delta_P}}{\partial z_i}(\mathbf{b}, \bar{\mathbf{b}}) t^{d-p_i} + (\text{Higher terms}), \tag{4.4}$$

$$\frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f^I_{\Delta_P}}{\partial \bar{z}_i}(\mathbf{b}, \bar{\mathbf{b}}) t^{d-p_i} + (\text{Higher terms}). \tag{4.5}$$

Consider above expansions and compare the order of t in both sides of Equation 4.1. Then we get

$$\overline{\frac{\partial f^I_{\Delta_P}}{\partial z_i}(\mathbf{b}, \bar{\mathbf{b}})} = \lambda_0 \frac{\partial f^I_{\Delta_P}}{\partial \bar{z}_i}(\mathbf{b}, \bar{\mathbf{b}}).$$

Hence $\mathbf{b} \in \mathbb{C}^{*m} \cap \operatorname{Sing} f^I_{\Delta_P}$, which is impossible.
(*b*) If $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \equiv 0$, this follows by the same method as in (*a*). □

Singularities Of Spherical Map

By Lemma A, there exists $r_0 > 0$ and $0 < \delta_0 \ll r_0$ such that for any $0 < r < r_0$ and $0 < \|\eta\| < \delta < \delta_0$ with $\delta \ll r$

$$f^{-1}(\eta) \pitchfork S_r^{2n-1}.$$

To prove the existence of Spherical Fibration, we have to show that the spherical map $\varphi := \frac{f}{\|f\|} = \mathrm{Re}(\sqrt{-1}\log(f))$ has no singular points over $S_r^{2n-1} \setminus K_r$, where $K_r = V(f) \cap S_r^{2n-1}$.

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Proposition 4.1

A point $\mathbf{z} \in \mathrm{Sing}\,\varphi$ if and only if $\exists \lambda \in \mathbb{R}$ such that

$$\sqrt{-1}(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{d}\log f(\mathbf{z},\bar{\mathbf{z}})) = \lambda \mathbf{z}.$$

Theorem B

Under the assumption of Theorem A, f has spherical fibration.

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Lemma B

If $f : \mathbb{C}^n \to \mathbb{C}$ is a strongly non-degenerate mixed polynomial, then $\exists r_0 > 0$ such that for any $0 < r < r_0$, the two vectors $\sqrt{-1}(\overline{d \log f}(\mathbf{z}, \bar{\mathbf{z}}) - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}))$ and $\mathbf{z}$ are linearly independent over $\mathbb{R}$ for any $\mathbf{z} \in S_r^{2n-1} \setminus K_r$.

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Under the assumption of Theorem A, f has spherical fibration.

Lemma B

If $f : \mathbb{C}^n \to \mathbb{C}$ is a strongly non-degenerate mixed polynomial, then $\exists r_0 > 0$ such that for any $0 < r < r_0$, the two vectors $\sqrt{-1}(\overline{d \log f(\mathbf{z}, \bar{\mathbf{z}})} - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}))$ and $\mathbf{z}$ are linearly independent over $\mathbb{R}$ for any $\mathbf{z} \in S_r^{2n-1} \setminus K_r$.

☞ The proof of Lemma B is similar in spirit to Lemma A. Note that $\mathrm{Sing}\,\varphi \subset \mathrm{Sing}\, f \setminus V(f)$, therefore we only require here the strong non-degeneracy of f.

By Theorem A and B, we have following fibrations

$$f : f^{-1}(S^1_\delta) \cap B^{2n}_r \to S^1_\delta \tag{4.6}$$

$$\varphi : S^{2n-1}_r \setminus K_r \to S^1 \tag{4.7}$$

for $0 < \delta \ll r$.

To show the equivalence of above fibrations, we adapt the method inflation of the tube. Let $\mathbf{z} \in f^{-1}(\eta), \eta \neq 0$ be a smooth point. Then the tangent space $T_{\mathbf{z}} f^{-1}(\eta)$ is a real subspace of $\mathbb{R}^{2n}$ whose vectors are orthogonal in $\mathbb{R}^{2n}$ to

$$v_1(\mathbf{z}) = \overline{d \log f}(\mathbf{z}, \bar{\mathbf{z}}) + \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}) \tag{4.8}$$

$$v_2(\mathbf{z}) = \sqrt{-1}(\overline{d \log f}(\mathbf{z}, \bar{\mathbf{z}}) - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}})). \tag{4.9}$$

☞ In holomorphic case, v_1 and v_2 are perpendicular.

Theorem C

Under the assumption of Theorem A, the Milnor tube fibration of f is isomorphic to the spherical fibration of φ.

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Lemma C

Assume f is a strongly non-degenerate mixed polynomial. Then there exists $r_0 > 0$ such that for any $\|\mathbf{z}\| < r_0$ with $f(\mathbf{z}, \bar{\mathbf{z}}) \neq 0$, the vectors $\mathbf{z}$, $v_1(\mathbf{z})$ and $v_2(\mathbf{z})$ satisfy one of the following conditions

1. They are linearly independent over $\mathbb{R}$;
2. There exists $a, b \in \mathbb{R}$ with $a > 0$ such that

$$\mathbf{z} = a v_1(\mathbf{z}) + b v_2(\mathbf{z}).$$

The Proof is based on the Milnor's arguments for holomorphic case. Let $N(K_r) := \{\mathbf{z} \in S_r^{2n-1} | \|f(\mathbf{z}, \overline{\mathbf{z}})\| \leqslant \delta\}$ be the neighborhood of K_r in S_r^{2n-1}. Since $N(K_r) \cong B_\delta \times K_r$ with $B_\delta = \{\eta \in \mathbb{C} | \|\eta\| \leq \delta\}$, the restriction $\varphi : S_r^{2n-1} \setminus \operatorname{Int} N(K_r) \to S^1$ is isomorphic to the spherical fibration.

By Lemma C and Partition Of Unity, there exists a vector field $\mathbf{v}(\mathbf{z})$ over $T = \{\mathbf{z} \in B_r^{2n-1} | \|f(\mathbf{z}, \overline{\mathbf{z}})\| \geq \delta\}$ such that

$$\begin{cases} \operatorname{Re}\langle \mathbf{v}(\mathbf{z}), \mathbf{z} \rangle > 0, \\ \operatorname{Re}\langle \mathbf{v}(\mathbf{z}), v_1(\mathbf{z}) \rangle > 0, \\ \operatorname{Re}\langle \mathbf{v}(\mathbf{z}), v_2(\mathbf{z}) \rangle = 0. \end{cases}$$

Pushing out along the trajectories of the vector field, we have

$$f^{-1}(S_\delta^1) \cap B_r^{2n} \cong S_r^{2n-1} \setminus \operatorname{Int} N(K_r).$$

Therefore we obtain the following commutative diagram

$$\begin{array}{ccc} f^{-1}(S^1_\delta) \cap B^{2n}_r & \xrightarrow{f} & S^1_\delta \\ \downarrow \psi & & \downarrow \frac{1}{\delta} \\ S^{2n-1}_r \setminus \operatorname{Int} N(K_r) & \xrightarrow{\varphi} & S^1 \end{array}$$

where ψ is the homeomorphism.
Therefore the two fibrations are isomorphic. $\square$

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where ψ is the homeomorphism.
Therefore the two fibrations are isomorphic. □

Remark 4.2

In case of isolated singularities, $S^{2n-1}_r \setminus K_r$ always fibers over S^1, even if the spherical fibration may not exist.

Comment

In the holomorphic setting, Némethi and Zaharia gave an effective estimation of the Bifurcation set for a non-degenerate polynomial. In the mixed setting, we can define Newton Boundary at infinity to obtain such kind of estimation. Moreover, it was shown that the bifurcation set of a strongly non-degenerate mixed polynomial is bounded, which implies the existence of the global monodromy fibration.

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Problem

- Can we find other examples(not necessary to be non-degenerate) with Milnor fibrations in the mixed setting?

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Problem

- Can we find other examples(not necessary to be non-degenerate) with Milnor fibrations in the mixed setting?
- Assume $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ are two holomorphic analytic germs. If $f\bar{g} : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at 0, does $f\bar{g}$ have Milnor tube fibration?

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Thanks!