# Milnor Fibration Of Mixed Polynomials(II)

## The Bridge Between Real And Complex Singularity Theory

Ying Chen

Department of Mathematics
HUST

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# OUTLINE

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1. Motivation
2. Resolution Of Mixed Singularities
3. Topology Of Mixed Curves

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2. Resolution Of Mixed Singularities

3. Topology Of Mixed Curves

4. Remarks and Questions

# Calculation Of Milnor Number

For a holomorphic polynomial germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularities, the Milnor number of $\mathbf{0}$ can be computed via the following formula

$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

If we denote the Milnor fiber of $f$ by $F$, then $\mu(f) = \dim H_{n-1}(F)$.

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## Kouchnirenko's Formula for $n = 2$

If $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is non-degenerate and convenient, then

$$\mu(f) = 2S - a - b + 1$$

where $S$ is the area of the compact polygon limited by $\Gamma(f)$, $(a, 0)$ and $(0, b)$ are the end-points of $\Gamma(f)$ on the coordinate axes.

**Problem**

Can we resolve singularities of a strongly non-degenerate and convenient mixed polynomial?

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[Oka, 2009] Consider the mixed curve $V = \{z_1\bar{z}_1 - z_2^2 = 0\}$. The toric modification is associated with the following regular fan generated by the vertices

$$\Sigma = \left\{ E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Take the cone $\sigma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then in $\mathbb{C}^2_\sigma$, the strict transformation $\hat{V}$ is not transversal to the exceptional divisor $E(P)$.

Go back

# Blow Up In $\mathbb{C}$

- (Real Blowing Up) Define $\tau_R : \mathbb{C}^* \to \mathbb{C} \times \mathbb{RP}^1$ by

$$z = x + \sqrt{-1}y \mapsto (z, [x : y]).$$

Consider $\mathrm{R}\mathbb{C} \overset{\text{def}}{=} \overline{\mathrm{Im}\,\tau_R}$ with charts $(U_0, (\tilde{x}, t))$ and $(U_1, (\tilde{y}, s))$.
Note that $\tilde{x} = x, t = \frac{y}{x}$ and $\tilde{y} = y, s = \frac{x}{y}$.
The canonical projection $\omega_{\mathrm{R}} : \mathrm{R}\mathbb{C} \to \mathbb{C}$ is

$$\omega_{\mathrm{R}}(\tilde{x}, t) = \tilde{x}(1 + \sqrt{-1}t), \qquad \omega_{\mathrm{R}}(\tilde{y}, s) = \tilde{y}(1 + \sqrt{-1}s).$$

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- (Polar Blowing Up) Define $\tau_P : \mathbb{C}^* \to \mathbb{R}^+ \times S^1$ by

$$z = r\exp(i\theta) \mapsto (r,\ \exp(i\theta)).$$

Consider $\mathrm{P}\mathbb{C} \stackrel{\text{def}}{=} \mathbb{R}^+ \times S^1$ with the canonical projection

$$\omega_{\mathrm{P}} : \mathrm{P}\mathbb{C} \to \mathbb{C}, \qquad \omega_{\mathrm{P}}(r, \exp(i\theta)) = r\exp(i\theta).$$

# Blowing Up Along Exception Divisors I

Given a $n$-dimensional complex manifold $X$ with finite many smooth divisors $E_1, \ldots, E_k$ such that the union $E = \cup_{i=1}^k E_i$ has at most normal crossing singularities, we use real blowing up on the normal 1-dimensional subspaces of $E_i$ for $1 \leq i \leq k$. This processes is called normal real blowing up along divisors. Let us denote it by $\omega_{\mathrm{R}} : \mathrm{R}X \to X$. Hence we have

1. $\mathrm{R}X$ is a smooth manifold and $\omega_{\mathrm{R}} : \mathrm{R}X \setminus \omega_{\mathrm{R}}^{-1}(E) \to X \setminus E$ is a diffeomorphism.
2. For $1 \leq i \leq k$, $\tilde{E}_i \overset{\text{def}}{=} \omega_{\mathrm{R}}^{-1}(E_i)$ is a real codimension 1 variety. Let $E_I \overset{\text{def}}{=} \cap_{i \in I} E_i \setminus \cup_{j \notin I} E_j$ and $\tilde{E}_I \overset{\text{def}}{=} \omega_{\mathrm{R}}^{-1}(E_I)$. Then $\omega_{\mathrm{R}}| : \tilde{E}_I \to E_I$ is a fibration with fiber $S^k$ for $k = |I|$.

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☞Choosing a local coordinate, can you describe $\tilde{E}_I$?

# Blowing Up Along Exception Divisors II

We can also consider normal polar blowing up along divisors which is denoted by $\omega_{\mathrm{P}} : \mathrm{P}X \to X$. Here $\mathrm{P}X$ is a smooth manifold with boundary.

Let us take $Q \in E_1$ and choose a local coordinate $(U, (u_1, \ldots, u_n))$ such that $E_1 = \{u_1 = 0\}$. Then

$$\omega_{\mathrm{P}}^{-1}(U) \cong \mathrm{P}\mathbb{C} \times \mathbb{C}^{n-1}$$

with coordinates $(r_1, \exp(i\theta_1), u_2, \ldots, u_n)$ satisfying

$$(r_1, \exp(i\theta_1), u_2, \ldots, u_n) \mapsto (u_1, \ldots, u_n),$$

where $u_1 = r_1 \exp(i\theta_1)$. Therefore $\omega_{\mathrm{P}}^{-1}(E_1)$ is given by $\{r_1 = 0\}$.

## Blowing Up Along Exception Divisors II

We can also consider normal polar blowing up along divisors which is denoted by $\omega_P : PX \to X$. Here $PX$ is a smooth manifold with boundary.
Let us take $Q \in E_1$ and choose a local coordinate $(U, (u_1, \dots, u_n))$ such that $E_1 = \{u_1 = 0\}$. Then

$$\omega_P^{-1}(U) \cong P\mathbb{C} \times \mathbb{C}^{n-1}$$

with coordinates $(r_1, \exp(i\theta_1), u_2, \dots, u_n)$ satisfying

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where $u_1 = r_1 \exp(i\theta_1)$. Therefore $\omega_P^{-1}(E_1)$ is given by $\{r_1 = 0\}$.

☞Find the relation between the two blowing up along divisors.

# Good Resolution

Assume that $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\upsilon,\mu} c_{\upsilon,\mu} \mathbf{z}^{\upsilon} \bar{\mathbf{z}}^{\mu}$ is a strongly non-degenerate and convenient mixed polynomial with $c_{0,0} = 0$.

# GOOD RESOLUTION

Assume that $f(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{\upsilon,\mu} c_{\upsilon,\mu} \mathbf{z}^{\upsilon} \overline{\mathbf{z}}^{\mu}$ is a strongly non-degenerate and convenient mixed polynomial with $c_{0,0} = 0$.

**Main Theorem**

There exists a good resolution $\Phi$, such that the multiplicity of $\Phi^*(f)$ along each exceptional divisors $\tilde{E}(P)$ is $d(P)$.

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**Main Theorem**

There exists a good resolution $\Phi$, such that the multiplicity of $\Phi^*(f)$ along each exceptional divisors $\tilde{E}(P)$ is $d(P)$.

Firstly, we take the toric modification $\hat{\pi} : X \to \mathbb{C}^n$ associated with the regular simplicial cone subdivision $\Sigma^*$ of dual Newton polyhedron $\Gamma^*(f)$. Let $\mathrm{V}^+$ be the set of strictly positive vertices in $\Sigma^*$. Then taking the polar blowing up along the exception divisors $E = \cup E(P)$ for $P \in \mathrm{V}^+$, we have

$$\Phi := \hat{\pi} \circ \omega_{\mathrm{P}} : \mathrm{P}X \xrightarrow{\omega_{\mathrm{P}}} X \xrightarrow{\hat{\pi}} \mathbb{C}^n.$$

with $\tilde{E}(P) := \omega_{\mathrm{P}}^{-1}(E(P))$. The map $\Phi$ is the desired resolution.

## Example

Let us consider ▸ Example.

Take the toric chart $(u_1, u_2)$ of $\mathbb{C}^2_\sigma$. The strict transformation can be written as follows

$$\hat{V} = \{(u_1, u_2) \in \mathbb{C}^{*2}_\sigma | \bar{u}_1 - u_1 u_2^2 = 0\}$$

with the exceptional divisor $E(P) = \{u_1 = 0\}$.
By using polar blowing up, we have

$$\tilde{V} = \{(r_1, \theta_1, u_2) \in \mathrm{P}\mathbb{C} \times \mathbb{C} | u_2 \pm \exp(-i\theta_1) = 0\}$$

where $\tilde{E}(P) = \{r_1 = 0\}$. Note that the multiplicity of

$$\Phi^*(f) = r_1^2[1 - \exp(2i\theta_1)u_2^2]$$

along $\tilde{E}(P)$ is $2$ which coincides with $d(P)$.

# An Important Example

Consider a familly of mixed polynomials

$$f_t(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 + t z_1 \bar{z}_1^2 - z_2^3$$

with $t \in \mathbb{C}$. Then $f_t$ is strongly non degenerate iff $\|t\| \neq 1$.
Denote the plane curve germ of $C_t = V(f_t)$ by $(C_t, 0)$ and the numbers of their link components by $\mathrm{lkn}(C_t, 0)$.

1. For $0 \leq \|t\| < 1$, the link of $(C_t, 0)$ are equivalent to the link of $(C_0, 0)$. In this case, $\mathrm{lkn}(C_t, 0) = 3$.
2. For $\|t\| > 1$, the link of $(C_t, 0)$ are equivalent to the link of $(C^{'}, 0)$ for $C^{'} := \{z_1 \bar{z}_1^2 - z_2^3 = 0\}$. In this case, $\mathrm{lkn}(C_t, 0) = 1$.

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☞In holomorphic case, for a curve $C$ defined by a non-degenerate convenient polynomial, the number of its link components $\mathrm{lkn}(C, 0)$ is completely determined by Newton boundary.

# CONDITIONS OF FACE FUNCTIONS

- A vertex $A = (a, b) \in \Gamma(f)$ is called simple if the face function $f_A$ contains only a single monomial $z_1^{a_1} z_2^{b_1} \bar{z}_1^{a_2} \bar{z}_2^{b_2}$ with $a = a_1 + a_2$ and $b = b_1 + b_2$. Otherwise, $A$ is called a multiple vertex. In particular, we say $\Gamma(f)$ is simple if all its vertices are simple.

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- For a monomial $z_1^{a_1} z_2^{b_1} \bar{z}_1^{a_2} \bar{z}_2^{b_2}$ with $a_1 + a_2 > 0$ and $b_1 + b_2 > 0$, we say it is polar admissible if $a_1 \neq a_2$ and $b_1 \neq b_2$. In this case, the vertex $A = (a, b)$ is not on the coordinate axe.

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**Remark 3.1**

1. Consider a good resolution $\Phi$ for $f$ of Main theorem. If a vertex $A \in \Gamma(f)$ is simple, then $\Phi^*(f)$ does not intersect with $\tilde{E}(A)$.
2. The polar admissible monimial was introduced by M.Oka to investigate non-isolated mixed singularities under the super non-degeneracy.

# GOOD NEWTON POLAR BOUNDARY

For $\Delta \in \Gamma(f)$, $f_\Delta$ is called a good polar weighted polynomial if its polar weights coincide with its radial weights, and moreover

$$f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = c\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{n}} \prod_{j=1}^{k} (z_2^a \bar{z}_2^{a'} - \lambda_j z_1^b \bar{z}_1^{b'})^{\mu_j} \tag{†}$$

with $a \neq a'$, $b \neq b'$, $\lambda_j \neq 0$ and $\gcd(a, a', b, b') = 1$. Note that $f_\Delta$ is non-degenerate iff $\mu_j = 1$, for all $1 \leq j \leq k$.
If every face function is good polar weighted, then we say that $f$ has a good Newton polar boundary.

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### Remark 3.2

In the above factorization $(\dagger)$, we say that $g_j = z_2^a \bar{z}_2^{a'} - \lambda_j z_1^b \bar{z}_1^{b'}$ is an irreducible binomial polar weighted polynomial.

## Link Components

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a mixed polynomial and $C = f^{-1}(0)$. We denote the number of link components which are not contained in the coordinate axes by $\mathrm{lkn}^*(C, 0)$.

**Theorem D**

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a strongly non-degenerate convenient polynomial. If $\Gamma(f)$ is simple, then we have

$$\mathrm{lkn}(C, 0) = \sum_{\Delta \in \mathcal{F}} \mathrm{lkn}^*(f_\Delta^{-1}(0), 0)$$

where $\mathcal{F}$ is the set of 1-faces of $\Gamma(f)$.

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where $\mathcal{F}$ is the set of 1-faces of $\Gamma(f)$.

Given a non-degenerate good polar weighted polynomial $f_\Delta$ with the factorization $(\dagger)$, we have

$$\mathrm{lkn}^*(f_\Delta^{-1}(0), 0) = k\gcd(a - a', b - b').$$

# Zeta Function Of The Monodromy (I)

Consider a non-degenerate and convenient mixed polynomial $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ which is polar weighted and radial weighted homogenous of radial type $(q_1, q_2; d_r)$ and polar type $(p_1, p_2; d_p)$.

Let us denote the Milnor fibre of $f$ by $F$ and $F^* \stackrel{\text{def}}{=} F \cap \mathbb{C}^{*2}$.
The monodromy map $h : F^* \to F^*$ is

$$h(z_1, z_2) = (z_1 \omega^{p_1}, z_2 \omega^{p_2}), \qquad \omega = \exp(\frac{2\pi i}{d_p}).$$

Suppose that the face functions of the edge vertices in $\Gamma(f)$ are $z_1^{a_1} \|z_1\|^{2b_1}$ and $z_2^{a_2} \|z_2\|^{2b_2}$, the zeta function of $h : F \to F$ is

$$\zeta(t) = \frac{(1 - t^{d_p})^{-\chi(F^*)/d_p}}{(1 - t^{a_1})(1 - t^{a_2})}, \qquad ((\divideontimes))$$

where $\chi(F^*)$ is the Euler Characteristic of $F^*$.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a non-degenerate and convenient mixed polynomial and $\Delta_1, \ldots, \Delta_k$ be the 1-faces of $\Gamma(f)$. We assume

## ZETA FUNCTION OF THE MONODROMY (II)

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a non-degenerate and convenient mixed polynomial and $\Delta_1, \ldots, \Delta_k$ be the 1-faces of $\Gamma(f)$. We assume

**Assumption***

- Every face function $f_{\Delta_j}$ is *PWH* of type $(p_{j1}, p_{j2}; d_{p_j})$.
- the face functions of the simple vertices $M_j = \Delta_j \cap \Delta_{j+1}$ are polar admissible for $1 \le j \le k-1$. Moreover, the monomials of the vertices on the coordinate axes are $z_1^{a_1}\|z_1\|^{2b_1}$ and $z_2^{a_2}\|z_2\|^{2b_2}$.

# Zeta Function Of The Monodromy (II)

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## Theorem E

Under Assumption*, the zeta function of $h : F \to F$ is

$$\zeta(t) = \frac{\prod_{j=1}^{k} (1 - t^{d_{p_j}})^{-\chi(F_j^*)/d_{p_j}}}{(1 - t^{a_1})(1 - t^{a_2})}.$$

**Connectivity Theorem(M.Oka, 2018)**

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a mixed polynomial satisfying Milnor Conditions $(A)$ and $(B)$. Take a Milnor radius $r_0$ and assume that there exists a mixed smooth point $w \in V(f)$ with $\|w\| < r_0$ and the sphere of radius $\|w\|$ intersect $V(f)$ transversely at $w$. If $\mathrm{codim}_{\mathbb{R}} V(f) = 2$, then the Milnor fibre is connected.

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**Remark 3.3**

- Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a strongly non-degenerate and convenient mixed polynomial. Then $f$ has an isolated singularity at the origin, by above theorem, the Milnor fibre is connected.

# Connectivity Of Fibre

## Connectivity Theorem(M.Oka, 2018)

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## Remark 3.3

- Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a strongly non-degenerate and convenient mixed polynomial. Then $f$ has an isolated singularity at the origin, by above theorem, the Milnor fibre is connected.
- Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a polar weighted homogenous mixed polynomial with $\operatorname{codim}_{\mathbb{R}} V(f) = 2$. If we assume that $V(f)$ has a smooth point, then the Milnor fibre is connected.

# Euler Characteristic Of Fibre

Let $f_\Delta : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a non-degenerate and convenient mixed polynomial of polar type $(p_1, p_2; d_p)$ with $\Delta = \overline{AB}$.
Suppose that the vertices $A$ and $B$ on the coordinate axes are simple and polar admissible. Denote the characteristic polynomials of the monodromy map for $i$-th homology group $H_i(F_\Delta)$ by $P_i(t)$, where $i = 0, 1$. We have $P_0(t) = 1 - t$ and $\zeta(t) = P_0^{-1}(t)P_1(t)$.
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The Wang Sequence of Milnor fibration is

$$0 \to H_2(S^3_\varepsilon \setminus K_\varepsilon) \to H_1(F_\Delta) \stackrel{h_*-id}{\to} H_1(F_\Delta) \to H_1(S^3_\varepsilon \setminus K_\varepsilon) \to \mathbb{Z} \to 0.$$

Let $f_\Delta : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a non-degenerate and convenient mixed polynomial of polar type $(p_1, p_2; d_p)$ with $\Delta = \overline{AB}$.
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The Wang Sequence of Milnor fibration is

$$0 \to H_2(S^3_\varepsilon \setminus K_\varepsilon) \to H_1(F_\Delta) \overset{h_* - id}{\to} H_1(F_\Delta) \to H_1(S^3_\varepsilon \setminus K_\varepsilon) \to \mathbb{Z} \to 0.$$

Write $l_\Delta = \mathrm{lkn}^*(f_\Delta^{-1}(0))$ and $\mu_\Delta$ for the multiplicity of factor $(1 - t)$ in $P_1(t)$. Then by using formular $(\divideontimes)$, we have

$$\mu_\Delta = l_\Delta - 1 = -\chi(F^*_\Delta)/d_p - 1.$$

## Kouchnirenko's Formula for mixed curves

Assume that $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a non-degenerate convenient mixed polynomial satisfying Assumption*. Write $l_j = \mathrm{lkn}^*(f_{\Delta_j}^{-1}(0))$ for $1 \leq j \leq k$. Then the Milnor number $\mu(f)$ is

$$\mu(f) = \sum_{j=1}^{k} l_j d_{p_j} - a_1 - a_2 + 1.$$

### Kouchnirenko's Formula for mixed curves

Assume that $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a non-degenerate convenient mixed polynomial satisfying Assumption*. Write $l_j = \mathrm{lkn}^*(f_{\Delta_j}^{-1}(0))$ for $1 \leq j \leq k$. Then the Milnor number $\mu(f)$ is

$$\mu(f) = \sum_{j=1}^{k} l_j d_{p_j} - a_1 - a_2 + 1.$$

### Example

Let $f = z_1^3\bar{z}_1 + z_2^2\bar{z}_2$. Then $f$ is a non-degenerate convenient mixed polynomial of polar type $(1,2;2)$. Here $a_1 = 2$, $a_2 = 1$ and $\mathrm{lkn}^*(f^{-1}(0)) = 2$. By using above formula, we have $\mu(f) = 2$. In fact, the Milnor fibre of $F$ is homotopic equivalent to $S^1 \vee S^1$.

## Comment

1. Theorem E also holds true for non-convenient case. If there is no end vertice on the coordinate axe $z_1$, then we can eliminate the term $(1 - t^{a_1})$ in the Zeta function.
2. In higher dimension, M.Oka generalize the formula of Zeta function for non-degenerate convenient mixed functions of strongly polar weighted homogeneous face type.

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## Problem

- Under the hypothesis of Connectivity Theorem, can we obtain the homotpic type of the Milnor fibre?
- Can we computer the Milnor number for degenerate mixed polynomials with isolated singularities?
- For a strongly non-degenerate convenient mixed polynomial in global sense, can we resolve the singularity at infinity?

# Thanks!