SELECTED EXERCISES BASED ON THE CIMPA 2021 TALKS

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1. Exercises for Blum Medial Axis for Medical Imaging

1) Let $\Omega \subset \mathbb{R}^2$ be a region with smooth boundary $\mathcal{B}$. For a point $y_0 \in \mathcal{B}$. let $\gamma(t)$ denote a unit speed parametrization of a neighborhood of $\gamma(0) = y_0$ in the counterclockwise direction.

i) For $u_0 \in int(\Omega)$, show that the distance squared function $f_{u_0}(y) = \|y - u_0\|^2$ has a singular point at y_0 iff $y_0 - u_0$ is orthogonal to $T_{y_0}\mathcal{B}$ (note that this is the same as for the distance function).

ii) Let $\mathbf{n}$ denote the unit normal at y_0 so that for $\mathbf{t} = \gamma'(0)$, $(\mathbf{t}, \mathbf{n})$ has positive orientation, so then $\mathbf{n}$ points inward. We can choose coordinates so y_0 is the origin and $\mathbf{t}$ points along the positive x-axis, and so $\mathbf{n}$ points along the positive y axis. Let $\kappa(t)$ denote the signed curvature of $\gamma(t)$ (so $\mathbf{t}' = \kappa(t)\cdot\mathbf{n}$).

Show that f_{u_0} has a degenerate critical point at y_0 iff $\|y_0 - u_0\| = \frac{1}{\kappa(0)}$. Thus, u_0 is the center of curvature (center of the osculating circle) for $\gamma(t)$ at the point y_0 (and hence is a point on the focal set of $\mathcal{B}$ for y_0. Hence, if u_0 is not a focal point then f_{u_0} has a nondegenerate critical point at y_0 and hence defines an A_1 point.

iii) If there is another point $y_1 \in \mathcal{B}$ such that f_{u_0} has a nondegenerate critical point at y_1, and $(f_{u_0}(y_0))^{\frac{1}{2}}$ is the minimum distance of u_0 to $\mathcal{B}$ which only happens at y_0 and y_1, then u_0 is an A_1^2 point of the Blum medial axis.

iv) Show that the first three derivatives of f_{u_0} at y_0 are zero iff it is a degenerate critical point at y_0 and $\kappa'(0) = 0$ (i.e. it is also a critical point for the curvature, i.e. it is a "vertex point").

v) in the case of iii) show that the sign of t^4 in the Taylor expansion of $f_{u_0}(\gamma(t))$ at 0 is given by $-\kappa''(0)$. If this is nonzero then $f_{u_0}(\gamma(t))$ has an A_3 singularity at 0. If the osculating circle lies within Ω then u_0 is an edge point of the Blum medial axis.

vi) In the case of v), let (v_1, v_2) denote local coordinates around u_0 with u_0 the origin. Show that $F_{u_0}(\gamma(t), (v_1, v_2)) = \|\gamma(t) - (u_0 + (v_1, v_2)\|^2$ is the versal unfolding of the germ $f_{u_0}(\gamma(t)) = \|\gamma(t) - u_0\|^2$ at 0.

(Hint: use the special coordinates from ii) and the form of the Taylor expansion of $f_{u_0}(\gamma(t))$ at $t = 0$ resulting from the previous steps.)

vii) Try to carry out the computation for a higher dimensional Ω, e.g. in $\mathbb{R}^3$ so $\mathcal{B}$ is then a smooth surface, using instead an appropriate Monge patch for the surface centered at y_0.

2) For the Blum medial structure (M, U) for a generic region $\Omega \subset \mathbb{R}^{n+1}$ with boundary $\mathcal{B}$, the radial function satisfies the inequality

$$r \; < \; \frac{1}{\kappa_{r\,i}}\,, \text{for all positive principal radial curvatures } \kappa_{r\,i}\,.$$

Using this and the formula relating the differential geometric shape operator and the radial shape operator show the following:

i) the formula given in the lecture relating the principal curvatures of $\mathcal{B}$ and principal radial curvatures is valid and implies (using the above inequality) that they have the same signs;
ii) From i) deduce that for $\mathbb{R}^3$, radial umbilic points (where the principal radial curvatures agree) correspond to umbilic points of $\mathcal{B}$.
iii) Show that the principal radial directions (eigenvectors of the radial shape operator) correspond to the principal directions on $\mathcal{B}$.

(Hint: see the proof of [Thm 3.2, D2]).

2. Exercises for Singularity Theory for Natural Images

These exercises concern the abstract mappings appearing in the classifications.

3) First we consider the models for projection mappings from a smooth surface with local germs $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. These require some use of the Malgrange Preparation Theorem.

i) Show that the fold and cusp map germs

$$f(x, y) = (x^2, y) \qquad \text{and} \qquad g(x, y) = (x^3 + yx, y)$$

are infinitesimally stable, i.e. their $\mathcal{A}_e$-codimensions are 0.

ii) Show that the "lips-beaks" germs $h(x, y) = (x^3 + \varepsilon y^2 x, y)$, with $\varepsilon = \pm 1$, have $\mathcal{A}_e$-codimensions 1 with $(x, 0)$ spanning $N\mathcal{A}_e \cdot h$, i.e. a complement to $T\mathcal{A}_e \cdot h$. Thus,

$$H(x, y, u) \; = \; (x^3 + \varepsilon y^2 x + ux, y, u)$$

is the $\mathcal{A}$-versal unfolding of $h(x, y)$.

iii) In ii) by examining the critical set and its image, determine which ε gives the "lips" and which gives the "beaks".

Second, we consider the germs of mappings under ${}_{\mathcal{V}}\mathcal{A}$-equivalence for several simple examples of stratifications $\mathcal{V}$.

4) Let $\mathcal{V}$ denote the y-axis, which could represent an edge, marking curve, one sheet of a ridge crease, or a fold shade or cast shadow curve.

i) Show that the germ $f(x, y) = (y, x^2 + yx)$ has ${}_{\mathcal{V}}\mathcal{A}_e$-codimension 0 and so is infinitesimally stable for ${}_{\mathcal{V}}\mathcal{A}$-equivalence.
ii) Show that the germ $g(x, y) = (y, x^3 + yx)$ has ${}_{\mathcal{V}}\mathcal{A}_e$-codimension 1 and give a term spanning the complement $T_{\mathcal{V}}\mathcal{A}_e \cdot g$ and the resulting ${}_{\mathcal{V}}\mathcal{A}$-versal unfolding.

(Hint: for more details see e.g. [BG2]).

5) Let $\mathcal{V}$ denote the corner formed by the upper right quadrant in $\mathbb{R}^2$ with boundary consisting of the positive x and y axes.

i) The analytic closure of the boundary consists of the union of the x and y axes. The module of analytic vector fields tangent to this analytic set is generated by $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$.

ii) Let $\rho(x)$ denote the infinitely flat smooth function

$$\rho(x) \;=\; \begin{cases} \exp(-\frac{1}{x^2}) & x < 0, \\ 0 & x \geq 0 \end{cases}$$

Then, verify that $\rho(x)\frac{\partial}{\partial y}$is tangent to the strata of $\mathcal{V}$; but it is not in the module of smooth vector fields generated by the vector fields in i).

iii) By the results of [DGH], this stratification is an example of a "special semi-analytic stratification" so the module of smooth vector fields tangent to the stratification $\mathcal{V}$ differs from the module of smooth vector fields generated by the two vector fields $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$ in i) by infinitely flat vector fields. This form is necessary by the example in ii).

iv) Consider the fold map germ $f(x, y) = (x^2, y)$ composed with the rotation given by the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$ to give

$$g(x, y) \;=\; ((ax + by)^2, -bx + ay) = (z_1, z_2)\,.$$

The fold critical curve of g is the line $ax+by = 0$. The rotation depends on the angle of this line, and there are two distinct cases depending whether the line intersects the first quadrant or not and also two cases depending on which side of the line the folding occurs. As we vary the angle, we obtain a one parameter family of germs. Suppose the line is transverse to both axes, so both $a, b \neq 0$. Show you can change coordinates in source, preserving $\mathcal{V}$, and target so that the germ is ${}_{\mathcal{V}}\mathcal{A}$-equivalent to $g_2(x, y) = (cx^2 + y^2, x + \varepsilon y)$, where $\varepsilon = \pm 1$ and $c \neq 0$ is a continuous parameter.

Hint: Begin by changing coordinates by adding z_2^2 to z_1 in the target to remove the xy term in the first coordinate. You may then linearly change coordinates in the source by multiplying x and y by positive constants (and possibly by reflecting about the line $x = y$), as well as making further linear changes of coordinates in the target to obtain

$$g_2(x, y) \;=\; (cx^2 + y^2, x + \varepsilon y), \qquad c \neq 0,\, \varepsilon = \pm 1$$

Note this is the normal form for this germ given for L_{2V} in [Chap. 6, DGH], and see [Ta].

v) Show that for a fixed $c \neq 0$, g_2 has ${}_{\mathcal{V}}\mathcal{A}_e$-codimension 2, with $N_{\mathcal{V}}\mathcal{A}_e \cdot g_2$ spanned by $(x^2, 0)$ and $(x, 0)$. By the versal unfolding theorem, for fixed c, a ${}_{\mathcal{V}}\mathcal{A}$-versal unfolding is given by

$$G_2(x, y, u) \;=\; ((c + v)x^2 + y^2 + ux, y, u, v)\,.$$

vi) Note

$$G_3(x, y, u) \;=\; (cx^2 + y^2 + ux, y, u)\,.$$

is weighted homogeneous, and it can be shown that the unfolding has finite ${}_{\mathcal{V}}\mathcal{A}$-codimension as an unfolding of g_2. Hence, by a topological versality theorem for Thom-Mather theory (see [Thm 9.10, D8a] and [Sect 4., D8b]), any further unfoldings by terms of "non-negative weight" i.e. weight $\geq$ 2 in the first entry, and weight ≥ 1 in the second are topologically ${}_{\mathcal{V}}\mathcal{A}$-equivalent to the trivial unfolding of G_3. Thus, G_3 is a topologically ${}_{\mathcal{V}}\mathcal{A}$-versal unfolding of g_2. The unfolding by G_3 has the effect of moving the fold critical line off the origin. Slightly changing the angle of the fold critical line does not change the topological properties of the unfolding.

vii) Consequently show the topological classification of the topologically versal unfolding is then determined by $\varepsilon \cdot sign(c)$. Determine the possibilities taking into account visibility.

References

The references given refer to the those in the select bibliography.