Hypersurfaces

Algebraic methods and singularities

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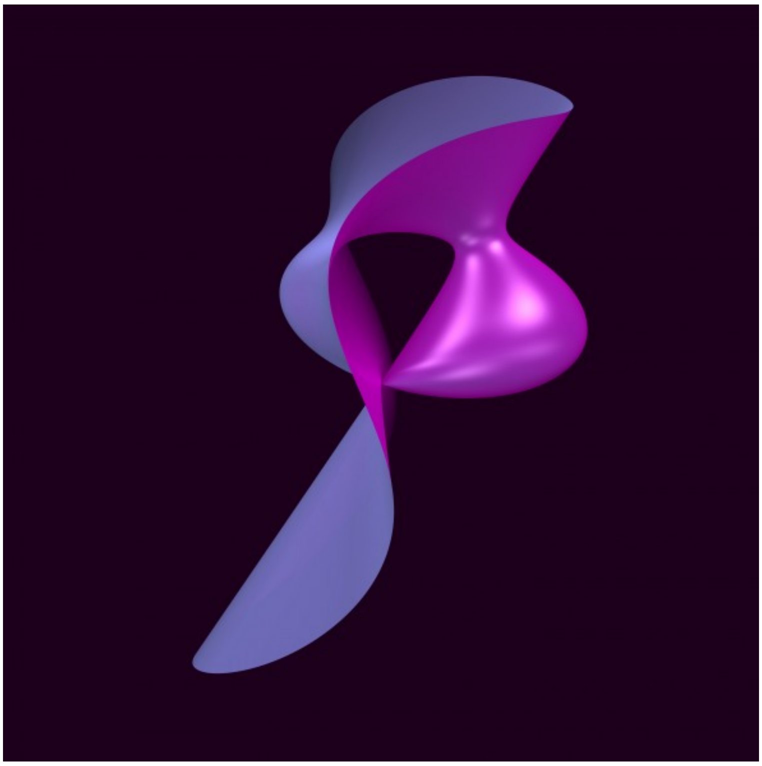

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Lecture 1

Chapter 1

Basics on classification of singularities

Classification of objects is a very important issue in mathematics. We have well known examples in differential topology, like classification of connected, orientable surfaces by genus, and in linear algebra, as classification of quadratic forms. In calculus, we have that a non degenerate critical point of a smooth function of two variables (real case) is either a local maximum, a local minimum or a saddle point. This is a simple case of Morse's Lemma. Our main subject in this course is classification of degenerate critical points of a smooth functions as well as their invariants.

The bibliography for chapters 1 and 2 is C.G. Gibson, Singular points of smooth mappings [12].

1.1 Germs and Jets

Let N and P be real or complex manifolds of dimensions n and p respectively, and $x \in N$. In the set of smooth (C^∞ in real case or holomorphic in complex case) mappings defined in a neighborhood of x in N into P we introduce the following equivalence relation:

Definition 1.1.1. Two mappings $f_1 : U_1 \to P$ and $f_2 : U_2 \to P$ are equivalent if there exists a neighborhood $U \subset U_1 \cap U_2$ of x in N such that $f_1 \mid_U$ and $f_2 \mid_U$ coincide.

An equivalence class under this relation is called *germ of mapping or map-germ* at x and is denoted by $f : (N, x) \to (P, y)$, $y = f(x)$. An element of an equivalence class is called *representative of the germ*.

Let $f : (N, x) \to (P, y)$ be a map-germ at x. Its *derivative* $df_x : T_xN \to T_yP$ is defined as the derivative at x of any representative of the germ. We say that $f : (N, x) \to (P, y)$ is a *germ of a diffeomorphism* if one of its representative (and so any) is a local diffeomorphism. It follows from the Inverse Function Theorem that a map-germ at x is a germ of a diffeomorphism if and only if its derivative at x is an isomorphism.

Let $f : (N, x) \to (P, y)$ and $g : (P, y) \to (M, z)$ be two map-germs where M is a manifold and $z \in M$. We define the composition $g \circ f : (N, x) \to (M, z)$ as: take representatives $\tilde{f} : U \to P$ and $\tilde{g} : V \to M$, $\tilde{f}(U) \subset V$, of f and g respectively, the map-germ $g \circ f$ is the equivalence class of $\tilde{g} \circ \tilde{f}$.

The *rank* of a map-germ at x is defined as the rank of its derivative at x. When the rank is n the map-germ is *immersive* and when the rank is p the map-germ is *submersive*. We say that the map-germ is *singular* when it is neither immersive nor submersive.

Definition 1.1.2. Two map-germs $f_1 : (N_1, x_1) \to (P_1, y_1)$ and $f_2 : (N_2, x_2) \to (P_2, y_2)$ are $\mathscr{A}$-equivalent if there exist germs of diffeomorphisms $h : (N_2, x_2) \to (N_1, x_1)$ and $k : (P_2, y_2) \to (P_1, y_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} (N_1, x_1) & \xrightarrow{f_1} & (P_1, y_1) \\ h \uparrow & & k \uparrow \\ (N_2, x_2) & \xrightarrow{f_2} & (P_2, y_2) \end{array}$$

that is, $f_1 = k \circ f_2 \circ h^{-1}$ (or $f_1 \circ h = k \circ f_2$).

Note that since any map-germ $f : (N, x) \to (P, y)$ is $\mathscr{A}$-equivalent to some germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we consider only smooth map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$.

Definition 1.1.3. (a) The **jet space** $J^k(n, p)$ is the $\mathbb{K}$-vector space of mappings $f : \mathbb{K}^n \to \mathbb{K}^p$, $f = (f_1, \ldots, f_p)$, where each component f_i, $i = 1, \ldots, p$, is a polynomial of degree $\leq k$ in the coordinates $x_1, x_2, \ldots, x_n$ of $\mathbb{K}^n$ with zero constant term. The elements of $J^k(n, p)$ are called k**-jets**.

(b) Let $U \subset \mathbb{K}^n$ be an open and convex subset. For each mapping $f : U \to \mathbb{K}$ we define the mapping $j^k f : U \to J^k(n, 1)$ as: $j^k f(a)$ is the k-th order Taylor polynomial of $f(x + a) - f(a)$ at the origin for all $a \in U$. We say that $j^k f(a)$ is the k-jet of f at a and $j^k f$ is the k-jet extension of f.

(c) Since $J^k(n, p) = J^k(n, 1) \times \ldots \times J^k(n, 1)$ (p-times), for $f : U \to \mathbb{K}^p$, $f = (f_1, \ldots, f_p)$, we define the mapping $j^k f : U \to J^k(n, p)$ as $j^k f(a) = (j^k f_1(a), \ldots, j^k f_p(a))$, for all $a \in U$.

Example 1.1.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ then

$$j^k f(a) = df_a(x) + \frac{1}{2!} d^2 f_a(x, x) + \cdots + \frac{1}{k!} d^k f_a(x, \ldots, x).$$

We denote by $\mathscr{O}_{n,p}$ the set of smooth map-germs $f : (\mathbb{K}^n, 0) \to \mathbb{K}^p$. When $p = 1$ we denote it by $\mathscr{O}_n$.

Exercises

1. Show that $\mathscr{O}_n$ is a local ring whose maximal ideal is $\mathfrak{m}_n = \{f \in \mathscr{O}_n \,/\, f(0) = 0\}$.

2. Show that $\mathscr{O}_n$ is a $\mathbb{K}$-vector space. Therefore $\mathscr{O}_n$ is a $\mathbb{K}$-algebra.

3. Show that $\mathscr{O}_{n,p}$ is an $\mathscr{O}_n$-module of rank p.

4. Show that $f \in \mathscr{O}_n$, $f(0) \neq 0$ has a multiplicative inverse in $\mathscr{O}_n$.

1.2 Lie group and Lie action

Definition 1.2.1. Let G be a commutative group with identity and let X be a set. An action of G on X is a map $G \times X \to X$ denoted by $(g, x) \mapsto g \cdot x$ such that:

(i) $1 \cdot x = x$ for all $x \in X$, where 1 is the identity element of G.

(ii) $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$, $x \in X$.

Definition 1.2.2. For x an element of X, the *orbit* of x is the subset $G \cdot x = \{g \cdot x \,/\, g \in G\}$ of X.

An action induces the equivalence relation: $x, y \in X$ are equivalent if there exists $g \in G$ such that $y = g \cdot x$. The orbit of x is its equivalence class under this relation.

Definition 1.2.3. A Lie group G is a manifold endowed with a group structure such that the group operations are smooth. More concretely, the multiplication map $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$, and the inversion map $G \to G$, $g \mapsto g^{-1}$, are required to be smooth.

One important example of a Lie group is the general linear group $GL_n(\mathbb{K})$. The elements of this group can be seen either as linear isomorphisms $\mathbb{K}^n \to \mathbb{K}^n$ or non-singular $n \times n$ matrices. In this last way $GL_n(\mathbb{K})$ is an open subset of the vector space of all $n \times n$ matrices $M_n(\mathbb{K})$, therefore a manifold. Since the entries of the product of two matrices is a quadratic polynomial on the entries of the two matrices, and the entries of inverse of a non-singular matrix is a rational function on the entries of the matrix, $GL_n(\mathbb{K})$ is a Lie group.

Definition 1.2.4. An action of a Lie group G on a manifold M is an action $G \times M \to M$ that is a smooth mapping.

Let $H^d(n, p)$ be the $\mathbb{K}$-vector subspace of $J^d(n, p)$ of d-jets $f = (f_1, \ldots, f_p)$, where each component f_i, $i = 1, \ldots, p$, is a homogeneous polynomial of degree d. The natural action of $GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ on $H^d(n, p)$ is a Lie group action:

$$(GL_n(\mathbb{K}) \times GL_p(\mathbb{K})) \times H^d(n, p) \to H^d(n, p),$$

$$((h, k), f) \mapsto k \circ f \circ h^{-1}.$$

The orbits of a Lie group action are immersed submanifolds and in general they are not submanifolds.

Proposition 1.2.5. *Let $\varphi\colon G \times M \to M$ be an action of a Lie group G on a manifold M. Suppose the orbits are submanifolds of M, then for all $x \in M$ the mapping $\varphi_x\colon G \to G \cdot x$, $\varphi_x(g) = g \cdot x$, is a submersion.*

Proof. We first show that φ_x has constant rank at any $h \in G$ by showing that it coincides with the rank of φ_x at the identity element 1 of G.

Let $\theta : G \to G$ be the diffeomorphism given by $\theta(g) = hg$ and let $\psi : M \to M$ be the diffeomorphism given by $\psi(y) = h \cdot y$. Consider the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\varphi_x} & G \cdot x \\ \theta \downarrow & & \downarrow \psi \\ G & \xrightarrow{\varphi_x} & G \cdot x \end{array}$$

By the chain Rule the following diagram is commutative:

$$\begin{array}{ccc} T_1G & \stackrel{d(\varphi_x)_1}{\longrightarrow} & T_xG\cdot x \\ d\theta_1 \downarrow & & \downarrow d\psi_x \\ T_hG & \stackrel{d(\varphi_x)_h}{\longrightarrow} & T_{h\cdot x}G\cdot x \end{array}$$

Therefore the rank of φ_x at h coincides with the rank of φ_x at 1, for all $h \in G$.

The result now follows from Sard's Theorem that guarantees the existence of a regular value in the image of φ_x. □

Corollary 1.2.6. *Let $\varphi\colon G \times M \to M$ be an action of a Lie group G on a manifold M. Suppose the orbits are submanifolds of M. Then for all $x \in M$, $T_xG\cdot x = d(\varphi_x)_1(T_1G)$. We denote $T_xG\cdot x$ by $L\,G\cdot x$.*

In this course we are interested in the case $p = 1$.

Consider the Lie action of $GL_n(\mathbb{K})$ on $H^d(n,1)$: $\varphi : GL_n(\mathbb{K}) \times H^d(n,1) \to H^d(n,1)$, $\varphi(h, f) = f \circ h^{-1}$.

We want to obtain the tangent spaces to the orbits of this action: $L\,GL_n(\mathbb{K})\cdot f = d(\varphi_f)_1(M_n(\mathbb{K}))$, where $f \in H^d(n,1)$. Note first that φ_f is the composition:

$$\begin{array}{ccccc} GL_n(\mathbb{K}) & \longrightarrow & GL_n(\mathbb{K}) & \stackrel{\phi}{\longrightarrow} & GL_n(\mathbb{K})\cdot f \\ h & \mapsto & h^{-1} & \mapsto & f\circ h^{-1} \end{array}.$$

Therefore, $L\,GL_n(\mathbb{K})\cdot f = d(\varphi_f)_1(M_n(\mathbb{K})) = d\phi_1(M_n(\mathbb{K}))$. Let $\Delta_{ij} = (a_{kl}) \in M_n(\mathbb{K})$ such that $a_{kl} = 0$ for either $k \neq i$ or $l \neq j$ and $a_{ij} = 1$. Let $\varepsilon > 0$ be a small enough real number and $\gamma_{ij} : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{K})$, $\gamma_{ij}(t) = 1 + t\Delta_{ij}$. Then $d\phi_1(M_n(\mathbb{K}))$ is generated by $\{(\phi\circ\gamma_{ij})'(0)\}$. Now

$$\phi\circ\gamma_{ij}(t)(x_1,\dots,x_n) = f\circ(1+t\Delta_{ij})(x_1,\dots,x_n) = f(x_1,\dots,x_i+tx_j,\dots,x_n).$$

$$(\phi\circ\gamma_{ij})'(t) = \frac{d}{dt}f(x_1,\dots,x_i+tx_j,\dots,x_n) = \frac{\partial f}{\partial x_i}(x_1,\dots,x_i+tx_j,\dots,x_n).x_j.$$

$$(\phi\circ\gamma_{ij})'(0) = \frac{\partial f}{\partial x_i}(x_1,\dots,x_n).x_j.$$

Therefore $L\,GL_n(\mathbb{K})\cdot f$ is the vector subespace generated by $\frac{\partial f}{\partial x_i}.x_j$, $i,j = 1,\dots,n$, that is,

$$LGL_n(\mathbb{K})\cdot f = \mathbb{K}\{\frac{\partial f}{\partial x_i}.x_j\}_{1\le i,j\le n}.$$

Theorem 1.2.7 (Mather's Lemma)**.** *Let the Lie group G act smoothly on the manifold M, and suppose that the orbits are submanifolds. Let $N \subset M$ be a submanifold that has the following properties:*

(a) for all $y \in N$, $L\,G\cdot y \supset T_yN$;

(b) the dimension of $G\cdot y$ is independent of the choice of $y \in N$;

(c) N is connected.

Then N is contained in a single G orbit.

Proof. We are going to prove that $G \cdot y \cap N$ is an open subset of N for all $y \in N$. Now the result follows since distinct orbits are disjoint sets and N is connected.

Let $k = \dim L\,G \cdot y$ for all $y \in N$.

Let $\phi : G \times N \to M$ be given by $\phi(g, y) = g \cdot y$.

We claim that ϕ has constant rank at any $(g, y) \in G \times N$. First note that

$$\begin{aligned} d\phi_{(1,y)}(T_1 G \times T_y N) &= d\phi_{(1,y)}(T_1 G \times \{0\}) + d\phi_{(1,y)}(\{0\} \times T_y N) \\ &= L\,G \cdot y + T_y N = L\,G \cdot y. \end{aligned}$$

So, ϕ has constant rank at $(1, y)$, for all $y \in N$.

Let $g_0 \in G$ and consider the following diagram:

$$\begin{array}{ccc} G \times N & \xrightarrow{\phi} & M \\ \downarrow \alpha & & \downarrow \beta \\ G \times N & \xrightarrow{\phi} & M \end{array}$$

where $\alpha(g, y) = (gg_0, y)$ and $\beta(x) = g_0 \cdot x$ are diffeomorphisms. We have

$$\dim d\phi_{(g,y)}(T_{(g,y)} G \times N) = \dim d\phi_{(gg_0,y)}(T_{(gg_0,y)} G \times N)$$

for all $g \in G$ and for all $y \in N$. Taking $g_0 = g^{-1}$ we obtain $\dim d\phi_{(g,y)}(T_{(g,y)} G \times N) = k$, that is, ϕ has constant rank at any $(g, y) \in G \times N$.

Let $z \in G \cdot y \cap N$. There exists $g \in G$ such that $z = g \cdot y$. Let $\theta : U \to G \times N$ be a parameterization of a neighbourhood of (g, y) in $G \times N$, $\theta(0) = (g, y)$, $0 \in U$. Also let $\psi : V \to M$ be a parameterization of a neighbourhood of z in M, $\psi(0) = z$, $0 \in V$. By shrinking U and V if necessary we have that $\psi^{-1} \circ \phi \circ \theta$ has constant rank. By the Rank Theorem there exist coordinates $(x_1, \ldots, x_l)$ such that $\psi^{-1} \circ \phi \circ \theta(x_1, \ldots, x_l) = (x_1, \ldots, x_k, 0, \ldots, 0)$. Therefore, $\phi(\theta(U))$ is a submanifold of M of dimension k and $z \in \phi(\theta(U))$. As $\dim G \cdot y = k$, $G \cdot y$ and $\phi(\theta(U))$ coincide in a neighborhood of z. Since $N \subset \phi(G \times N)$ there exists a neighborhood of z in N contained in $\phi(\theta(U))$. Therefore there exists a neighborhood of z in N contained in $G \cdot y \cap N$. □

In what follows we want to obtain the orbits of the Lie action $GL_n(\mathbb{K}) \times H^d(n, 1) \to H^d(n, 1)$, $(h, f) \mapsto f \circ h^{-1}$.

1. When $d = 1$, $H^1(n, 1)$ is the vector space of linear forms in n variables. So there exist two orbits, one consisting of the 'null linear form and the other consisting of non zero linear forms.

2. When $d = 2$, $H^2(n, 1)$ is the vector space of quadratic forms in n variables. So after appropriate linear change of coordinates $f \in H^2(n, 1)$ can be written as $f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_s^2 - x_{s+1}^2 - \ldots - x_r^2$. The numbers s and r are called rank and index of f respectively. Notice that the rank of f is the rank of the hessian of f at the origin. By the Sylvester's law of inertia the rank is an invariant by linear change of coordinates but the index can vary (for example multiplying the form by -1). So we consider the semi-index defined as $\min(s, r - s)$. So the quadratic forms are classified by the rank and the semi-index. For $n = 2$ and $f(x, y) = ax^2 + 2bxy + cy^2$ we have:

- $f = 0$ has rank 0 and is called of symbolic type.
- f has rank 1 and is called of parabolic type: $b^2 = ac$.
- f has rank 2 and we have two orbits in the real case (one in the complex case): $b^2 - ac > 0$ of semi-index 1, called hyperbolic type and $b^2 - ac < 0$ of semi-index 0, called elliptic type. We call them non-degenerated in complex case.

Now we want to obtain the orbits in $H^3(2,1)$, that is, cubics in two variables. Let $p \in H^3(2,1)$ be a non-zero cubic form. We can always factor p over $\mathbb{C}$ as $p = \ell_1\ell_2\ell_3$, where ℓ_i are linear forms in $\mathbb{C}[x,y]$, $i = 1,2,3$. When $\mathbb{K} = \mathbb{R}$, either the three ℓ_i are real or one is real and the other two are complex conjugate.

Proposition 1.2.8. *Let $p \in H^3(2,1)$ be a non-zero cubic form which factors over $\mathbb{C}$ as $p = \ell_1\ell_2\ell_3$. We have:*

1. *Assume $\mathbb{K} = \mathbb{C}$, if ℓ_1, ℓ_2, ℓ_3 are pairwise non-collinear, then p is $Gl_2(\mathbb{C})$-equivalent to $x^2y + y^3$ (non-degenerate cubic).*
2. *Assume $\mathbb{K} = \mathbb{R}$, if ℓ_1, ℓ_2, ℓ_3 are pairwise non-collinear and real, then p is $Gl_2(\mathbb{R})$-equivalent to $x^2y - y^3$ (non-degenerate hyperbolic cubic).*
3. *Assume $\mathbb{K} = \mathbb{R}$, if ℓ_1, ℓ_2, ℓ_3 are pairwise non-collinear and only one is real, then p is $Gl_2(\mathbb{R})$-equivalent to $x^2y + y^3$ (non-degenerate elliptic cubic).*
4. *If ℓ_1, ℓ_2 are not collinear but ℓ_1, ℓ_3 are collinear, then p is $Gl_2(\mathbb{K})$-equivalent to x^2y (parabolic cubic).*
5. *If ℓ_1, ℓ_2, ℓ_3 are all collinear, then p is $Gl_2(\mathbb{K})$-equivalent to x^3 (symbolic cubic).*

Proof. We can always assume, up to $Gl_2(\mathbb{K})$-equivalence, that $\ell_1 = x$. In item 5, we have $\ell_2 = ax$ and $\ell_2 = bx$, for some $a, b \in \mathbb{K} \setminus \{0\}$. Thus, $p = abx^3$ which is obviously $Gl_2(\mathbb{K})$-equivalent to x^3.

In item 4, ℓ_1, ℓ_2 are not collinear (and real in the case $\mathbb{K} = \mathbb{R}$). We can assume, up to $Gl_2(\mathbb{K})$-equivalence, that $\ell_1 = x$ and $\ell_2 = y$. Since ℓ_1, ℓ_3 are collinear, $\ell_3 = ax$, for some $a \in \mathbb{K} \setminus \{0\}$. Thus, $p = ax^2y$ which is obviously $Gl_2(\mathbb{K})$-equivalent to x^2y.

Suppose now that ℓ_1, ℓ_2, ℓ_3 are pairwise non-collinear. When $\mathbb{K} = \mathbb{C}$ or when $\mathbb{K} = \mathbb{R}$ and the three are real, we can assume $\ell_1 = x$ and $\ell_2 = y$ and hence $p = xy(ax + by)$, for some $a, b \in \mathbb{K} \setminus \{0\}$. Taking a change $(x,y) \mapsto (x/a, y/b)$ and eliminating the constants, we get that p is $Gl_2(\mathbb{K})$-equivalent to $xy(x+y)$ which is also $Gl_2(\mathbb{K})$-equivalent to $x^3 - xy^2$. Moreover, when $\mathbb{K} = \mathbb{C}$ this can be converted into $x^3 + xy^2$. This shows items 1 and 2.

Finally, for item 3 we have $\mathbb{K} = \mathbb{R}$ but ℓ_2, ℓ_3 are complex conjugated. The quadratic form $\ell_2\ell_3$ is definite, so it can be normalised into $\pm(x^2 + y^2)$. Hence, p is $Gl_2(\mathbb{K})$-equivalent to $(ax + by)(x^2 + y^2)$, for some $a, b \in \mathbb{R}$, with $a^2 + b^2 > 0$. By taking a rotation of angle $-\theta$, where $a = \rho\cos\theta$ and $b = \rho\sin\theta$ and eliminating the constants, p is $Gl_2(\mathbb{K})$-equivalent to $x^3 + xy^2$. □

1.3 The algebra of germs of functions

We have seen that the set $\mathscr{O}_n$ of smooth map-germs $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ is a local ring whose maximal ideal is $\mathfrak{m}_n = \{f \in \mathscr{O}_n \,/\, f(0) = 0\}$. We want to characterize this maximal ideal in terms of the coordinates $x_1, \ldots, x_n$ of $\mathbb{K}^n$.

Theorem 1.3.1 (Hadamard's Lemma)**.** *Let U be an open ball centered at the origin in $\mathbb{K}^n$ and let $f : U \times \mathbb{K}^q \to \mathbb{K}$ be a smooth function such that $f(0, y) = 0$, for all $y \in \mathbb{K}^q$. Then there exist smooth functions $f_1, \ldots, f_n : U \times \mathbb{K}^q \to \mathbb{K}$ such that $f = x_1 f_1 + \cdots + x_n f_n$.*

Proof. Let $x = (x_1, \ldots, x_n)$. We have

$$f(x, y) = \int_0^1 \frac{d}{dt} f(tx, y)\, dt = \int_0^1 \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(tx, y) x_i\, dt = \sum_{i=0}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx, y)\, dt.$$

Therefore, $f(x, y) = \sum_{i=0}^{n} x_i f_i$, where $f_i(x, y) = \displaystyle\int_0^1 \frac{\partial f}{\partial x_i}(tx, y)\, dt$. □

Corollary 1.3.2. *(a) $\mathfrak{m}_n$ is the ideal generated by $x_1, \ldots, x_n$, that is, $\mathfrak{m}_n = \langle x_1, \ldots, x_n \rangle$.*

(b) $\mathfrak{m}_n^k = \{f \in \mathscr{O}_n \,/\, j^{k-1} f(0) = 0\}$ is the ideal generated by monomials of degree k in $x_1, \ldots, x_n$.

Let $\mathbb{K}[[x_1, \ldots, x_n]]$ be the ring of formal power series in $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$. Define $\varphi : \mathscr{O}_n \to \mathbb{K}[[x_1, \ldots, x_n]]$ as $\varphi(f) = \hat{f}$, where $\hat{f}$ is its Taylor series around 0.

Proposition 1.3.3. *(a) φ is a surjective $\mathbb{K}$−algebra homomorphism.*

(b) The kernel of φ is $\ker \varphi = \cap_{k=1}^{\infty} \mathfrak{m}_n^k := \mathfrak{m}_n^{\infty}$.

(c) When $\mathbb{K} = \mathbb{C}$, $\mathfrak{m}_n^{\infty} = \{0\}$. When $\mathbb{K} = \mathbb{R}$, $\mathfrak{m}_n^{\infty} \neq \{0\}$ and any $f \in \mathfrak{m}_n^{\infty}$ is called a flat function.

Proof. (a) The surjectivity is trivial in the complex case and in the real case it follows from Borel's Lemma. (c) Let $f : (\mathbb{R}, 0) \to \mathbb{R}$ be given by $f(x) = exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. We have that $f \in \mathfrak{m}_n^{\infty}$ and $f \neq 0$. □

When $\mathbb{K} = \mathbb{C}$, $\mathscr{O}_n$ is a Noetherian ring. This is not true over $\mathbb{R}$. To prove this we need the following well known result.

Theorem 1.3.4 (Nakayama's Lemma)**.** *Let R be a commutative ring with identity 1 and $\mathcal{M} \subset R$ an ideal such that $1 + x$ is an unity for all $x \in \mathcal{M}$. Let M be an R-module, $A, B \subset M$ be R-submodules, A finitely generated. If $A \subset B + \mathcal{M}A$ then $A \subset B$.*

Remark 1.3.5. If R is a local ring with maximal ideal $\mathcal{M}$ then R and $\mathcal{M}$ satisfies the conditions of Nakayama's Lemma.

We shall prove that when $\mathbb{K} = \mathbb{R}$, $\mathfrak{m}_n^{\infty}$ is not a finitely generated ideal, therefore $\mathscr{O}_n$ is not a Noetherian ring. Suppose $\mathfrak{m}_n^{\infty}$ is a finitely generated ideal. As $\mathfrak{m}_n^{\infty} \subset \{0\} + \mathfrak{m}_n \mathfrak{m}_n^{\infty}$ it follows from Nakayama's Lemma that $\mathfrak{m}_n^{\infty} \subset \{0\}$, contradicting Proposition 1.3.3.

Let M be an $\mathscr{O}_n$-module and $I \subset M$ a submodule. Notice that M is a $\mathbb{K}$-vector space and I is a subspace. We denote by $\dim_{\mathbb{K}} M/I$ the dimension of M/I as $\mathbb{K}$-vector space.

Proposition 1.3.6. *Let M be a free $\mathcal{O}_n$-module of finite rank. Then $\dim_{\mathbb{K}} M/I$ is finite if and only if there exists an integer $k \geq 1$ such that $\mathfrak{m}_n^k M \subset I$.*

Proof. Suppose $\dim_{\mathbb{K}} M/I$ is finite. Consider

$$I + M \supset I + \mathfrak{m}_n M \supset \ldots \supset I + \mathfrak{m}_n^k M \supset \ldots \supset I.$$

Therefore,

$$\dim_{\mathbb{K}} \frac{M}{I+M} \leq \dim_{\mathbb{K}} \frac{M}{I+\mathfrak{m}_n M} \leq \ldots \leq \dim_{\mathbb{K}} \frac{M}{I} \leq \infty.$$

Then there exists an integer $k \geq 1$ such that $I + \mathfrak{m}_n^k M = I + \mathfrak{m}_n^{k+1} M$, that is, $\mathfrak{m}_n^k M \subset I + \mathfrak{m}_n^{k+1} M$. Now it follows from Nakayama's Lemma that $\mathfrak{m}_n^k M \subset I$.

Suppose now that there exists an integer $k \geq 1$ such that $\mathfrak{m}_n^k M \subset I$. Notice that M is isomorphic to $\mathcal{O}_n \times \ldots \times \mathcal{O}_n$ (finite copies) and $M/\mathfrak{m}_n^k M$ is isomorphic to $\dfrac{\mathcal{O}_n}{\mathfrak{m}_n^k} \times \ldots \times \dfrac{\mathcal{O}_n}{\mathfrak{m}_n^k}$. Therefore

$$\dim_{\mathbb{K}} M/I \leq \dim_{\mathbb{K}} M/\mathfrak{m}_n^k M \leq \infty.$$

□

Exercises

1. Prove part (b) of Corollary 1.3.2.

2. Show that $\varphi : \mathcal{O}_n \to \mathbb{K}[[x_1, \ldots, x_n]]$, $\varphi(f) = \hat{f}$, where $\hat{f}$ is its Taylor series around 0, is a $\mathbb{K}-$algebra homomorphism.

Exercises

1. Show that $\text{Diff}(\mathbb{K}^n, 0) = \{h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0) \,/\, h$ is a germ of diffeomorphism$\}$ is a group with the composition operation.

2. Let $G = \text{Diff}(\mathbb{K}^n, 0) \times \text{Diff}(\mathbb{K}^p, 0)$. Show that $G \times \mathcal{O}_{n,p} \to \mathcal{O}_{n,p}$, $((h, k), f) \mapsto k \circ f \circ h^{-1}$ is an action of G on $\mathcal{O}_{n,p}$.

3. Show that the action of $GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ on $H^d(n, p)$:

$$(GL_n(\mathbb{K}) \times GL_p(\mathbb{K})) \times H^d(n, p) \to H^d(n, p),$$

$$((h, k), f) \mapsto k \circ f \circ h^{-1}$$

is a Lie group action.

Chapter 2

Mather's Groups

2.1 Right equivalence

In section 1.1 we defined the ring $\mathcal{O}_n$ of smooth map-germs $f : (\mathbb{K}^n, 0) \to \mathbb{K}$. We also defined the group $\mathrm{Diff}(\mathbb{K}^n, 0) = \{h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0) \,/\, h$ is a germ of diffeomorphism$\}$ with the composition operation.

Definition 2.1.1. Let $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be germs of smooth mappings.

(a) We say that f, g are $\mathcal{R}$-equivalent (or right-equivalent) if there exists $h \in \mathrm{Diff}(\mathbb{K}^n, 0)$ such that $g = f \circ h^{-1}$.

(b) We set $\mathcal{R} = \mathrm{Diff}(\mathbb{K}^n, 0)$. $\mathcal{R}$ acts on $\mathcal{O}_n$ as $h \cdot f = f \circ h^{-1}$.

The group $\mathcal{R}$ is not a Lie group and $\mathcal{O}_n$ is not a smooth manifold. However, the space of jets $J^k(n, 1)$ is a finite dimensional vector space and so a smooth manifold. Also the group $\mathcal{R}^{(k)} := \{j^k h(0) \,/\, h \in \mathcal{R}\}$ is a Lie group and it acts on $J^k(n, 1)$ as $j^k h(0) \cdot j^k f(0) = j^k(f \circ h^{-1})(0)$. In what follows we want to calculate $L\mathcal{R}^{(k)} \cdot \sigma$ where $\sigma = j^k f(0) \in J^k(n, 1)$. Let $\varphi : \mathcal{R}^{(k)} \times J^k(n, 1) \to J^k(n, 1)$ be given by $\varphi(\psi, \sigma) = j^k(\sigma \circ \psi^{-1})(0)$. We proved that $L\mathcal{R}^{(k)} \cdot \sigma = d(\varphi_\sigma)_1(T_1\mathcal{R}^{(k)})$.

Remark 2.1.2. Since we are only interested in jets at 0 we use the notation $j^k f$ for $j^k f(0)$.

Proposition 2.1.3. *Let $\sigma \in J^k(n, 1)$. Then*

$$L\mathcal{R}^{(k)} \cdot \sigma = \{j^k(\sum_{i=1}^{n} \frac{\partial \sigma}{\partial x_i} h_i) \,/\, (h_1, \ldots, h_n) \in J^k(n, n)\}.$$

Proof. As in the homogeneous case, $L\mathcal{R}^{(k)} \cdot \sigma = d\phi_1(T_1\mathcal{R}^{(k)}) = d\phi_1(J^k(n, n))$ where $\phi(\psi) = j^k(\sigma \circ \psi)$.

Let $h = (h_1, \ldots, h_n) \in J^k(n, n)$ and consider the path $\lambda(t) = 1 + th \in \mathcal{R}^{(k)}$ for t small enough. Then $d\phi_1(h) = \dfrac{d(\phi \circ \lambda)}{dt}(0)$:

$$\frac{d(\phi \circ \lambda)}{dt}(0) = \frac{d}{dt} j^k(\sigma \circ (1 + th))|_{t=0} = j^k(\frac{d}{dt}(\sigma \circ (1 + th)))|_{t=0} = j^k(\sum_{i=1}^{n} \frac{\partial \sigma}{\partial x_i} h_i).$$

□

Definition 2.1.4. (a) Let $f \in \mathcal{O}_n$. The tangent space to the orbit $\mathcal{R} \cdot f$ is defined as

$$T\mathcal{R} \cdot f = \{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \,/\, h_i \in \mathfrak{m}_n, i = 1, \ldots, n\}.$$

(b) The ideal $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is called Jacobian ideal of f.

(c) $T\mathcal{R} \cdot f = \mathfrak{m}_n J(f) \subset \mathcal{O}_n$ is an ideal.

(d) The $\mathcal{R}$-codimension of f is the dimension as vector space of the quocient $\dfrac{\mathfrak{m}_n}{T\mathcal{R} \cdot f}$:

$$\mathcal{R} - \text{codim}(f) = \dim_{\mathbb{K}} \frac{\mathfrak{m}_n}{T\mathcal{R} \cdot f} = \dim_{\mathbb{K}} \frac{\mathfrak{m}_n}{\mathfrak{m}_n J(f)}.$$

In Lecture 4 we study some geometric aspects of the singularities. In order to do so we need the following definition.

Definition 2.1.5. (a) The extended tangent space is defined as $T\mathcal{R}_e \cdot f = J(f)$.
(b) The $\mathcal{R}_e$-codimension of f is

$$\mathcal{R}_e - \text{codim}(f) = \dim_{\mathbb{K}} \frac{\mathcal{O}_n}{T\mathcal{R}_e \cdot f} = \dim_{\mathbb{K}} \frac{\mathcal{O}_n}{J(f)}.$$

The $\mathcal{R}_e - \text{codim}(f)$ is also called Milnor number and denoted by $\mu(f)$.

Proposition 2.1.6. *Let $f \in \mathfrak{m}_n$. Then $\mathcal{R}$ − codim(f) is finite if and only if $\mathcal{R}_e$ − codim(f) is finite.*

Proof. Suppose that $\mathcal{R} - \text{codim}(f)$ is finite. Then

$$\dim_{\mathbb{K}} \frac{\mathcal{O}_n}{J(f)} \leq \dim_{\mathbb{K}} \frac{\mathcal{O}_n}{\mathfrak{m}_n J(f)} = 1 + \dim_{\mathbb{K}} \frac{\mathfrak{m}_n}{\mathfrak{m}_n J(f)}.$$

Therefore $\mathcal{R}_e - \text{codim}(f)$ is finite. On the other hand, since $\dim_{\mathbb{K}} \frac{\mathcal{O}_n}{J(f)}$ is finite it follows by Proposition 1.3.6 that there exists an integer k, $k \geq 1$ such that $\mathfrak{m}_n^k \subset J(f)$. Then $\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n J(f)$. Therefore again by Proposition 1.3.6 the $\mathcal{R} - \text{codim}(f)$ is finite. □

Corollary 2.1.7. *Let $f \in \mathfrak{m}_n$. Then $\mathcal{R}$ − codim(f) is finite if and only if there exists an integer $k > 0$ such that $J(f) \supset \mathfrak{m}_n^k$.*

Proposition 2.1.8. *Let $f \in \mathfrak{m}_n$ such that $\mathcal{R}_e$ − codim$(f) > 0$ and finite. Then the origin is an isolated singularity.*

Proof. Since $\dim_{\mathbb{K}} \mathcal{O}_n/J(f) > 0$ then $J(f) \subset \mathfrak{m}_n$ and therefore the origin is a singular point of f.

As $\dim_{\mathbb{K}} \mathcal{O}_n/J(f)$ is finite there exists an integer $k > 0$ such that $J(f) \supset \mathfrak{m}_n^k$. Then there exist u_{ij}, $1 \leq i, j \leq n$, such that $x_j^k = \sum_{i=1}^n u_{ij} \partial f / \partial x_i$, for all $j = 1, \ldots, n$. Let $P = (a_1, \ldots, a_n)$ be a singular point of f close enough to the origin. As $\frac{\partial f}{\partial x_i}(P) = 0$ then $a_j^k = 0$ and $P = 0$. □

The converse of above proposition is true only when $\mathbb{K} = \mathbb{C}$.

Example 2.1.9. Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ be given by $f(x, y) = (x^2 + y^2)^2$. The origin $(0, 0)$ is an isolated singularity but $\dim_{\mathbb{R}} \mathscr{O}_2 / J(f) = \infty$.

Proposition 2.1.10. *Let $f \in \mathfrak{m}_n$ such that $0 < \mathscr{R} - codim(f) < \infty$. Then*

$$\mathscr{R} - codim(f) = \mathscr{R}_e - codim(f) + n - 1.$$

Proof. We claim that $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}$ is a linearly independent sequence. In fact suppose that there exists $\xi = (\xi_1, \dots, \xi_n) \in \mathscr{O}_{n,n}$, $\xi(0) \neq 0$, such that $\sum_{i=1}^{n} \xi_i \dfrac{\partial f}{\partial x_i} = 0$. We can think of ξ as a vector field without singularity and so can be retified. Then there exists a germ of diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$, $h = (h_1, \dots, h_n)$, such that $\xi_i \circ h = \partial h_i / \partial x_1$. Let $g = f \circ h$. It is easy to see that $\mathscr{R} - \text{codim}(g)$ is finite. Besides,

$$\frac{\partial g}{\partial x_1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \circ h \, \frac{\partial h_i}{\partial x_1} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \circ h \; \xi_i \circ h = 0$$

Therefore the origin is a non-isolated singularity of g, contradicting the fact that the $\mathscr{R}-\text{codim}(g)$ is finite.

Then $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}$ is a linearly independent sequence and so $\dim_{\mathbb{K}} \mathbb{K}\{\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\} = n$ where $\mathbb{K}\{\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\}$ is the subespace of $\mathscr{O}_n$ generated by the partial derivatives of f.

Since $J(f) \simeq \mathfrak{m}_n J(f) \oplus \mathbb{K}\{\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\}$ we have

$$\dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f)} = \dim_{\mathbb{K}} \frac{\mathscr{O}_n}{\mathfrak{m}_n J(f)} - n = \dim_{\mathbb{K}} \frac{\mathfrak{m}_n}{\mathfrak{m}_n J(f)} + 1 - n.$$

□

Any map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ induces a homomorphism $f^* : \mathscr{O}_p \to \mathscr{O}_n$ in the natural way: $f^*(h) = h \circ f$.

The following result shows that the Milnor number is an invariant of the singularity.

Proposition 2.1.11. *Let $f, g \in \mathfrak{m}_n$ be $\mathscr{R}$-equivalent map-germs. Then $\mu(f) = \mu(g)$.*

Proof. There exists a germ of diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$, $h = (h_1, \dots, h_n)$, such that $g = f \circ h$. Then $h^* : \mathscr{O}_n \to \mathscr{O}_n$ is an isomorphism. We have

$$\frac{\partial g}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \circ h \, \frac{\partial h_j}{\partial x_i} = \sum_{j=1}^{n} h^*(\frac{\partial f}{\partial x_j}) \frac{\partial h_j}{\partial x_i}.$$

Then $\partial g / \partial x_i \in h^*(J(f))$ and $J(g) \subset h^*(J(f))$. Similarly, $J(f) \subset (h^{-1})^*(J(g)) = (h^*)^{-1}(J(g))$. Therefore $h^*(J(f)) = J(g)$. □

Exercises

1. Prove that $\mathscr{R}^{(k)}$ is a smooth manifold by showing that it is an open subset of $J^k(n,n)$ and conclude that it is a Lie group.

2. Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ be given by $f(x,y) = (x^2+y^2)^2$. Show that $\dim_{\mathbb{R}} \mathscr{O}_2/J(f) = \infty$.

3. Use Nakayama's Lemma to show that $\mathfrak{m}_n J(f) + \mathfrak{m}_n^{k+1} \supset \mathfrak{m}_n^k$ if and only if $\mathfrak{m}_n J(f) \supset \mathfrak{m}_n^k$.

2.2 Mather's groups

In this course we are interested in the group $\mathscr{R}$. But there are more four Mather's groups. In this section we define them.

Definition 2.2.1. Let $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be germs of smooth mappings.

(a) f, g are $\mathscr{A}$-equivalent (right-left equivalent) if there exists $(h,k) \in \text{Diff}(\mathbb{K}^n,0)\times\text{Diff}(\mathbb{K}^p,0)$ such that $g = k \circ f \circ h^{-1}$.

(b) We set $\mathscr{A} = \text{Diff}(\mathbb{K}^n,0) \times \text{Diff}(\mathbb{K}^p,0)$, the group of pair of diffeomorphisms. $\mathscr{A}$ acts on $\mathfrak{m}_n\mathscr{O}_{n,p}$ as $(h,k)\cdot f = k \circ f \circ h^{-1}$.

(c) f, g are $\mathscr{L}$-equivalent (left equivalent) if there exists $k \in \text{Diff}(\mathbb{K}^p,0)$ such that $g = k \circ f$.

(d) We set $\mathscr{L} = \text{Diff}(\mathbb{K}^p,0)$, the group of diffeomorphisms with the composition operation. $\mathscr{L}$ acts on $\mathfrak{m}_n\mathscr{O}_{n,p}$ as $k \cdot f = k \circ f$.

In order to study the geometry of the singularities we need to define two more equivalence relations.

Definition 2.2.2. Let $f, g \in \mathfrak{m}_n\mathscr{O}_{n,p}$.

(a) f, g are $\mathscr{C}$-equivalent if there exists a germ of diffeomorphism $H : (\mathbb{K}^n \times \mathbb{K}^p, 0) \to (\mathbb{K}^n\times\mathbb{K}^p, 0)$ such that $H(x,y) = (x, \theta(x,y))$, $\theta(x,0) = 0$, such that $H(x, f(x)) = (x, g(x))$. So H takes the graph of f into the graph of g.

(b) We set $\mathscr{C} = \{H \in \text{Diff}(\mathbb{K}^n \times \mathbb{K}^p, 0) \,/\, H(x,y) = (x,\theta(x,y)),\ \ \theta(x,0) = 0\}$. It acts on $\mathfrak{m}_n\mathscr{O}_{n,p}$ as $H \cdot f(x) = \theta(x, f(x))$.

(c) f, g are $\mathscr{K}$-equivalent (contact equivalent) if there exist germs of diffeomorphisms $h : (\mathbb{K}^n,0) \to (\mathbb{K}^n,0)$ and $H : (\mathbb{K}^n\times\mathbb{K}^p,0) \to (\mathbb{K}^n\times\mathbb{K}^p,0)$ such that $H(x,y) = (h(x), \theta(x,y))$, $\theta(x,0) = 0$, such that $H(x, f(x)) = (h(x), g(h(x)))$.

(d) We set $\mathscr{K} = \{(h,H) \in \text{Diff}(\mathbb{K}^n,0)\times\text{Diff}(\mathbb{K}^n\times\mathbb{K}^p,0) \,/\, H(x,y) = (h(x),\theta(x,y)),\ \ \theta(x,0) = 0\}$. It acts on $\mathfrak{m}_n\mathscr{O}_{n,p}$ as $H \cdot f(x) = \theta(h^{-1}(x), f(h^{-1}(x)))$.

Definition 2.2.3. The groups $\mathscr{R}, \mathscr{L}, \mathscr{A}, \mathscr{C}$ and $\mathscr{K}$ are called Mather's groups.

Similarly to the former section we can define the tangent spaces.

Definition 2.2.4. Let $f \in \mathfrak{m}_n\mathscr{O}_{n,p}$.

(a) The tangent space to the orbit $\mathscr{A}\cdot f$ is defined as

$$T\mathscr{A}\cdot f=\{\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\xi_i \,/\, \xi_i\in\mathfrak{m}_n, i=1,\dots,n\}+\{\eta\circ f \,/\, \eta\in\mathfrak{m}_p\mathscr{O}_{p,p}\}.$$

(b) The tangent space to the orbit $\mathscr{L}\cdot f$ is defined as

$$T\mathscr{L}\cdot f=\{\eta\circ f \,/\, \eta\in\mathfrak{m}_p\mathscr{O}_{p,p}\}.$$

(c) Let $I(f)$ be the ideal generated by the component functios of f, that is, $I(f)=\langle f_1,\dots,f_p\rangle$, $f=(f_1,\dots,f_p)$. The tangent space to the orbit $\mathscr{C}\cdot f$ is defined as

$$T\mathscr{C}\cdot f=I(f)\mathscr{O}_{n,p}.$$

(d) The tangent space to the orbit $\mathscr{K}\cdot f$ is defined as

$$T\mathscr{K}\cdot f=\{\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\xi_i \,/\, \xi_i\in\mathfrak{m}_n, i=1,\dots,n\}+I(f)\mathscr{O}_{n,p}.$$

We have seen that a map-germ $f:(\mathbb{K}^n,0)\to(\mathbb{K}^p,0)$ induces a homomorphism $f^*:\mathscr{O}_p\to\mathscr{O}_n$. Then we can write $I(f)=f^*(\mathfrak{m}_p)$ and $T\mathscr{C}\cdot f=f^*(\mathfrak{m}_p)\mathscr{O}_{n,p}$.

Remark 2.2.5. 1. In the definition above, if one consider $\xi=(\xi_1,\dots,\xi_n)\in\mathscr{O}_{n,n}$ and $\eta\in\mathscr{O}_{p,p}$ we have the $\mathscr{G}$-extended tangent spaces:

$$T\mathscr{A}_e\cdot f=\{\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\xi_i \,/\, \xi_i\in\mathscr{O}_n, i=1,\dots,n\}+\{\eta\circ f \,/\, \eta\in\mathscr{O}_{p,p}\}.$$

$$T\mathscr{K}_e\cdot f=\{\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\xi_i \,/\, \xi_i\in\mathscr{O}_n, i=1,\dots,n\}+f^*(\mathfrak{m}_p)\mathscr{O}_{n,p}.$$

2. $T\mathscr{C}\cdot f=T\mathscr{C}_e\cdot f$.

One importante invariant of the singularity is its codimension:

Definition 2.2.6. Let $\mathscr{G}$=$\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$ and $f\in\mathfrak{m}_n\mathscr{O}_{n,p}$.

(a) The $\mathscr{G}$-codimension of f is

$$\mathscr{G}-\text{codim}(f)=\dim_{\mathbb{K}}\frac{\mathfrak{m}_n\mathscr{O}_{n,p}}{T\mathscr{G}\cdot f}.$$

(b) The $\mathscr{G}_e$-codimension of f is

$$\mathscr{G}_e-\text{codim}(f)=\dim_{\mathbb{K}}\frac{\mathscr{O}_{n,p}}{T\mathscr{G}_e\cdot f}.$$

Exercises

1. Show that if $f,g:(\mathbb{K}^n,0)\to(\mathbb{K}^p,0)$ are $\mathscr{A}$-equivalent then they are $\mathscr{K}$-equivalent.

2. Show that the converse is not true by giving one example of f,g $\mathscr{K}$-equivalent but not $\mathscr{A}$-equivalent.

Part II

Lecture 2

Chapter 3

Finite determinacy

The analysis of the conditions for a map-germ to be finitely determined and of the degree of determinacy involves the most important of the local aspects of singularity theory.

– C.T.C.Wall [31]

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a smooth function germ, where smooth means C^∞ when $\mathbb{K} = \mathbb{R}$ or holomorphic when $\mathbb{K} = \mathbb{C}$. In this chapter we discuss finitely $\mathscr{R}$-determined function-germs. Finite determinacy has been an important subject in singularity theory for many decades and the bibliography in this topic is extensive. We refer in this lecture to the clear presentation (with examples) in Chapter 3 of the book of Mond and Nuño-Ballesteros [20].

With regard to results on necessary and sufficient conditions of finite determinacy and estimates of the order of determinacy for Mather's group $\mathscr{G}$, we refer to Mather [17], Gaffney [10, 11], du Plessis [7], Damon [5] and Du Plessis, Bruce and Wall [3]. The survey article by Terry Wall [31] is a complete account of the theory of finite determinacy for Mather's groups $\mathscr{G} = \mathscr{A}, \mathscr{R}, \mathscr{L}, \mathscr{K}$ and $\mathscr{C}$ until 1981.

3.1 Finitely determined function germs

René Thom formulated in [27] the problem of characterizing singularities determined by their jet of some order. The name *finitely determined germs* was later given by John Mather [17], who also gave necessary and sufficient conditions for finite determinacy. The theorems of $\mathscr{R}$-finite determinacy that we discuss here were also given by Jean-Claude Tougeron in [28, 29].

Definition 3.1.1. Let $\mathscr{G}$ be a group acting in the space of germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. We say that f is k *-determined* with respect to $\mathscr{G}$-equivalence if there exists a positive integer k such that for all $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ with $j^k g(0) = j^k f(0)$, it follows that $f \simeq_{\mathscr{G}} g$. We say that f is $\mathscr{G}$*-finitely determined* if f is k-determined for some k. The denomination $\mathscr{G}$*-finite* germs is also widely used.

In these notes, we only discuss finite $\mathscr{R}$-determinacy of function germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$. The following theorems give necessary and sufficient conditions for finite determinacy. This section is mainly devoted to discuss these results.

Theorem 3.1.2. (Sufficient condition for finite determinacy)
Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$.
If $\mathfrak{m}_n Jf \supset \mathfrak{m}_n^k$ *then* f *is* k*-determined for* $\mathscr{R}$*-equivalence.*

Theorem 3.1.3. (Necessary condition for finite determinacy)
Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$.
If f *is* k*- determined for* $\mathscr{R}$*-equivalence then* $\mathfrak{m}_n Jf \supset \mathfrak{m}_n^{k+1}$.

Example 3.1.4. A critical point x_0 of a C^∞-function $f : M \to \mathbb{R}$ is *non-degenerate* if the Hessian matrix of f at x_0 is non-degenerate, that is $\det(Hessian(f)(x_0)) \neq 0$. The Morse Lemma ([21]) states that in a neighbourhood of a non-degenerate critical point, f is $\mathscr{R}$-equivalent to

$$f(x_1, \ldots, x_n) = f(x_0) + \Sigma_{i=1}^r x_i^2 - \Sigma_{i=r+1}^n x_i^2$$

This follows from the following conditions:

(i) the critical point x_0 is non-degenerate if and only if $Jf = \mathfrak{m}_n$.

(ii) Condition (i) implies that $\mathfrak{m}_n Jf = \mathfrak{m}_n^2$, and by the sufficient condition in Theorem 3.1.2, it follows that f is 2-determined.

(iii) By linear changes of coordinates, we can reduce f to a sum of squares.

Example 3.1.5. The germ at the origin of $f(x, y) = x^2 y$ is not finitely $\mathscr{R}$-determined. In fact, for any $\ell > k > 3$, $f_k(x, y) = x^2 y + y^{k+1}$ and $f_\ell l(x, y) = x^2 y + y^{\ell+1}$ have the same k jet as f at the origin, but $\mu(f_k) \neq \mu(f_\ell)$, which implies that they are not $\mathscr{R}$ equivalent.

The successful approach to finite determinacy was inspired by the action of a Lie group on finite dimensional manifolds.

Our groups are not Lie groups, and our function spaces are not Banach manifolds. But, the solution to the problem of finding necessary and sufficient conditions for a germ $f \in \mathscr{O}_n$ to be finitely determined, consists in reducing our infinitesimal approach to jet spaces.

Suppose f is k-G-determined. Then, given $g \in \mathscr{O}_n$, $j^k g(0) = j^k f(0)$, the one-parameter family

$$F : (\mathbb{K}^n \times \mathbb{K}, 0 \times \mathbb{K}) \to (\mathbb{K}, 0)$$
$$(x, t) \mapsto F(x, t) = (1 - t) f(x) + t g(x)$$

has a constant k-jet $j^k F_t(0) = j^k f(0) + t j^k (g - f)(0) = j^k f(0)$.

We will identify F with a "line" L in $\mathscr{O}_n$. Now, f is k finitely determined if and only if L is contained in a unique orbit for every choice of g as above.

A sufficient condition is to find a 1-parameter family H_t of elements in $\mathscr{G}$ such that $H_0 = 1 \in \mathscr{G}$, $H_t(0) = 0$, $H_t \cdot f_t = f$, for any $t \in \mathbb{K}$.

These conditions say that the family F is $\mathscr{G}$*-trivial.* The next step is to search for an infinitesimal condition, giving an equivalent characterization of triviality in terms of vector fields.

This step, in principle, is not hard: the equation $H_t \cdot F_t = f$ implies that $\frac{\partial}{\partial t}(H_t \cdot F_t) = 0$ leading to the desired infinitesimal condition. The converse follows from integration of vector fields.

We call this result "the Thom-Levine lemma", which is stated and proved below

Definition 3.1.6. A 1-parameter family $F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}, 0)$, $F(x, 0) = f(x)$ is $\mathscr{R}$-trivial if there is a 1-parameter family of germs of diffeomorphisms $H_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$

$$\begin{aligned} H : (\mathbb{K}^n \times \mathbb{K}, 0) &\to (\mathbb{K}^n, 0) \\ (x, t) &\mapsto h(x, t) \end{aligned}$$

such that $H(x, 0) = x$, $H_t(0) = 0$, and

$$f_t \circ H_t = f.$$

Remark 3.1.7. We also use the notation $F'(x, t) = (F(x, t), t)$, $H'(x, t) = (H(x, t), t)$ for the corresponding 1-parameter unfoldings. In this notation F is $\mathscr{R}$-trivial if $F' \circ H' = f \times \mathrm{Id}_{\mathbb{R}}$.

The next result is known as the Thom-Levine lemma (see [17, 7, 20]).

Proposition 3.1.8. *Let $f \in \mathscr{O}_n$ and $F' : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K} \times \mathbb{K}, 0)$, $F'(x, t) = (F(x, t), t)$, $F(0, t) = 0$, $F(x, 0) = f(x)$, the germ at 0 of a 1-parameter unfolding of f. Then F' is $\mathscr{R}$-trivial if and only there exists vector field $V : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^n \times \mathbb{K}, 0)$ with $V(x, t) = -v(x, t) + \frac{\partial}{\partial t}$, $v(x, t) = \sum_{i=1}^n v_i(x, t) \frac{\partial}{\partial x_i}$, $v_i(0, t) = 0$ for $i = 1, \ldots, n$ such that*

$$\frac{\partial F}{\partial t}(x, t) = \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x_i}(x, t) \cdot v_i(x, t). \tag{3.1}$$

Proof. We give here an idea of the proof. The reader may consult Mond and Nuño-Ballesteros [20, p. 37] for a complete proof (see also Mather [17, p. 144], du Plessis [7, p. 174]).

If F is a trivial unfolding of f, $F \circ H = f \times 1_{\mathbb{R}}$ and then $\partial \cdot (F \circ H) = 0$ and we apply the chain rule to get (3.1).

Conversely, if condition (3.1) holds, we consider the system of differential equations in $(\mathbb{R}^n \times \mathbb{R}, 0)$:

$$\begin{aligned} \dot{x} &= v(x, t) \\ v(0, t) &= 0 \end{aligned} \tag{3.2}$$

We consider the flow $H(x, t)$ associated to the system 3.2 with initial condition $H(x, 0) = x$. Notice that $\frac{\partial H}{\partial t}(x, t) = v(H(x, t), t)$.

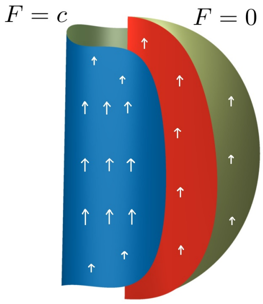


Figure: $\frac{\partial F}{\partial t} - \sum_{i=1}^n \frac{\partial \bar{f}}{\partial x_i} \cdot v_i = 0$

Now, at $x = 0$, $\frac{\partial H}{\partial t}(0,t) = v(H(0,t),t) = 0$, and since the null function is a solution of the equation, by uniqueness of solutions of 3.2 , it follows that $H(0,t) = 0$.

Then, we obtained a 1−parameter family H_t of diffeomorphisms of $(\mathbb{K}^n \times \mathbb{K}, 0)$ such that $H_0(x) = x$, $H_t(0) = 0$, such that $F_t \circ F_t = f$.

□

Condition (3.1) in Proposition 3.1.8 admits an useful algebraic formulation. First, we introduce some notation.

Given the 1-parameter unfolding $F' : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K} \times \mathbb{K}, 0)$, $F'(x,t) = (F(x,t),t)$ with $F(x,0) = f(x)$, as before, let

$$J_x F := \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle \tag{3.3}$$

be the $\mathscr{O}_{n+1}$ ideal generated by the partial derivatives of F with respect to the variables x_i, $i = 1, \ldots, n$.

Corollary 3.1.9. *Let $f \in \mathscr{O}_n$ and $F' : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K} \times \mathbb{K}, 0)$, $F'(x,t) = (F(x,t),t)$, $F(0,t) = 0$, $F(x,0) = f(x)$, the germ at 0 of a 1-parameter unfolding of f. Then F′ is $\mathscr{R}$-trivial if and only*

$$\frac{\partial F}{\partial t}(x,t) \in \mathfrak{m}_n J_x F \tag{3.4}$$

We are now prepared to prove Theorem 3.1.2.

Proof. (of Theorem 3.1.2): The sufficient condition

Let g be such that $j^k g(0) = j^k f(0)$ and $F(x,t) = (1-t)f(x) + tg(x),\ t \in \mathbb{K}$.

The idea of the proof is to show that for each $t = a \in \mathbb{K}$, the germ

$$F : (\mathbb{K}^n \times \mathbb{K}, 0 \times a) \to (\mathbb{K}, 0)$$

is $\mathscr{R}$ trivial.

We prove this assumption for $t = 0$, the other cases are similar. We first show that the hypothesis $\mathfrak{m}_n Jf \supset \mathfrak{m}_n^k$ implies equation 3.4.

Notice that

$$\frac{\partial F}{\partial x_i} - \frac{\partial f}{\partial x_i} = t\frac{\partial (g-f)}{\partial x_i} \Rightarrow (\frac{\partial F}{\partial x_i} - \frac{\partial f}{\partial x_i})x_j \in \mathfrak{m}_n^{k+1}\mathfrak{m}_t\mathscr{O}_{n+1} \tag{3.5}$$

and hence it follows that

$$\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n J_x F + \mathfrak{m}_n^{k+1}\mathfrak{m}_t\mathscr{O}_{n+1}.$$

We now apply Nakayama's lemma to get that

$$\mathfrak{m}_n^{k+1} \subset \mathfrak{m}_n J_x F.$$

As $\frac{\partial F}{\partial t} = g - f \in \mathfrak{m}_n^{k+1}$, we conclude that

$$\frac{\partial F}{\partial t} \in \mathfrak{m}_n J_x F,$$

and the result follows by corollary 3.1.9. Since the same argument holds in a neighbourhood of $t = a$, $\forall a \in [0,1]$, the proof is complete.

□

We give now some notation before proving theorem 3.1.3.

For every positive integer k, let

$$\mathscr{R}_k = \{h \in \mathscr{R} \mid j^k h(0) = Id_{\mathbb{K}^n}\},$$

where $Id_{\mathbb{K}^n}$ denotes the identity map in $\mathbb{K}^n$.

The action of $\mathscr{R}_k$ in $J^k(n,1)$ is trivial. The quotient group $\mathscr{R}^{(k)} = \mathscr{R}/\mathscr{R}_k$ acts on $J^k(n,1)$ as follows:

$$\begin{aligned} \mathscr{R}^{(k)} &\to J^k(n,1) \\ j^k h &\mapsto j^k h * j^k f = j^k(f \circ h) \end{aligned}$$

The *tangent space* of this action is defined by

$$T\mathscr{R}^{(k)}\sigma = \frac{\mathfrak{m}_n Jf + \mathfrak{m}_n^{k+1}}{\mathfrak{m}_n^{k+1}}, \tag{3.6}$$

where $\sigma = j^k f(0)$.

Proof. (of Theorem 3.1.3): The necessary condition.

Let f be $\mathscr{R}$ k-determined, so that

$$\mathscr{R}.f \supset \{g \in \mathscr{O}_n \mid j^k f(0) = j^k g(0)\},$$

and consequently their tangent spaces at f satisfy the relation

$$T\mathscr{R}.f \supset T\{g \in \mathscr{O}_n \mid j^k f(0) = j^k g(0)\},$$

Then, the projection of these spaces into $J^r(n,1)$, gives

$$T\mathscr{R}^{(r)}\sigma \supset \frac{\mathfrak{m}_n^{k+1}}{\mathfrak{m}_n^{r+1}},$$

or equivalently,

$$T\mathscr{R}.f + \mathfrak{m}_n^{r+1} \supset \mathfrak{m}_n^{k+1},$$

and the result follows by applying Nakayama's lemma.

□

Example 3.1.10. Let $f(x,y) = x^5 + y^5$. We have

$$T\mathscr{R}.f = \left\langle x^5, x^4 y, x y^4, y^5 \right\rangle \nsupseteq \mathfrak{m}_2^5,$$

and we apply theorem 3.1.3 to conclude that f is not 5-determined with respect to $\mathscr{R}$-equivalence.

We summarize in the next theorem the equivalent conditions to finite determinacy.

Theorem 3.1.11. (Infinitesimal criteria of finite determinacy) *Let $f \in \mathcal{O}_n$. The following conditions are equivalent.*

1. *f is finitely determined with respect to $\mathcal{R}$ equivalence*
2. *There exists an integer $k \geq 1$ such that $\mathfrak{m}_n^k \subset \mathfrak{m}_n Jf$*
3. *There exists an integer $r \geq 1$ such that $\mathfrak{m}_n^r \subset Jf$*
4. *$\mu(f) < \infty$*
5. *$\mathcal{R} - \operatorname{codim}(f) < \infty$*

Exercises

1. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, $f \in \mathfrak{m}_n^2$. Show that $\mu(f) = 1$ if and only if 0 is a non-degenerate critical point of f.
2. Suppose that $m_n Jg \supset \mathfrak{m}_n^{k+1}$ for all $g \in \epsilon(n)$ such that $j^k g(0) = j^k f(0)$. Show that f is $k - \mathcal{R}$-determineded.
3. Use 1. to show that $x^4 + y^4$ is 4-$\mathcal{R}$-determinaded.

3.2 The geometric criterium of finite determinacy

In this section we discuss the geometric meaning of finite determinacy. The following result is known as *geometric criterium of $\mathcal{R}$-determinacy.*

Proposition 3.2.1. *The following conditions hold for $f \in \mathcal{O}_n$.*

1. *If $\mu(f) < \infty$ then f has isolated critical point.*
2. *If $\mathbb{K} = \mathbb{C}$ and f has isolated critical point then $\mu(f) < \infty$.*

Proof. 1. Suppose $\mathfrak{m}_n^k \subseteq J(f)$ for some k. Then, $V(J(f)) \subseteq V(\mathfrak{m}_n^k) = \{0\}$, where $V(I)$ is the set germ in $(\mathbb{K}^n, 0)$ given by the zero locus of an ideal I. Hence, f has isolated critical point.

2. Assume $\mathbb{K} = \mathbb{C}$. If f has isolated critical point then $V(J(f)) = \{0\} = V(\mathfrak{m}_n)$. By the Rückert Nullstellensatz, $\sqrt{J(f)} = \sqrt{\mathfrak{m}_n} = \mathfrak{m}_n$ and hence $\mathfrak{m}_n^k \subseteq J(f)$ for some k. □

Example 3.2.2. Item 2 is not true when $\mathbb{K} = \mathbb{R}$. For instance, $f(x, y) = (x^2 + y^2)^2$ has isolated critical point but $\mu(f) = \infty$.

We saw in Theorem 3.1.11 that the Infinitesimal Criterion of $\mathcal{R}$-finite determinacy holds with the same proof whether we consider f as a real analytic, C^∞ or complex analytic map-germ.

The following result explains the relations among $\mathcal{R}$-finitely determined germs in these different local rings.

Proposition 3.2.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic map-germ. The following are equivalent*

(i) f is k-$\mathscr{R}$-determined in the space of real analytic function-germs.

(ii) f is k-$\mathscr{R}$-determined in the space of smooth function-germs.

(iii) The complexification of f, $f_{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, is k-$\mathscr{R}$-determined in the space of holomorphic function-germs.

Exercises

1. Prove Proposition 3.2.3.

2. Let $p(x, y)$ be a homogeneous polynomial of degree k in $\mathbb{C}[x, y]$. Show that p has isolated singularity at the the origin if and only if

$$p(x, y) = \prod_{i=1}^{k} (a_i x + b_i y),$$

where $(a_i, b_i) \neq (a_j, b_j)$, $i \neq j$, $i, j = 1, \ldots, k$.

3.3 The complete transversal

The main references for this section are [4] and [25].

The algorithm we give here is an useful tool for the classification of singularities. It is based on Poincaré's method of classifying a singularity, jet by jet. The general idea can be described as follows:

(1) Suppose we have determined a normal form for the k-jet of f, denoted $\sigma = j^k f$.

(2) Next we classify $\mathscr{R}^{k+1}$-orbits of all $k+1$-jets whose k-jet is σ.

(3) For each normal form in (2), we repeat the procedure replacing k by $k+1$, and inductively, for the sucessive jet spaces $k + j$, $j = 2, \ldots$. If f is finitely determined, the method converges after a finite number of steps.

We first give some notation. Let G be a Lie group acting smoothly in some affine space A and V_A the subjacent vector space. Let W be a vector subspace of V_A. We denote by LG the Lie algebra of the group G and by LGx the tangent space to the Gx at $x \in A$.

Theorem 3.3.1. (The complete transversal theorem) *Let G be a Lie group acting smoothly on the affine space A. Let $W \subset V_A$ be a vector subspace. If*

$$LG.(x + w) = LG.x, \quad \forall x \in A, \forall w \in W \tag{3.7}$$

$$\tag{3.8}$$

Then:

(a) $\forall x \in A,\ x + \{LG.x \cap W\} \subset G.x \cap \{x + W\}$

(b) Let $x_0 \in A$ *and* $T \subset W$ *a vector subspace such that* $LG.x_0 + T \supset W$*. Then for all* $w \in W$*,* $\exists g \in G$ *such that* $g(x_0 + w) = x_0 + t$ *for some* $t \in T$*.*

We first state the following corollary of Mather's lemma.

Lemma 3.3.2. *Let* $x \in A$ *and suppose that*

$$LG.x = LG.(x + w),\ \forall w \in W$$

If $W \subset LG.x$ *then* $x + W \subset G.x$

We leave the proof as an exercise to the reader.

Remark 3.3.3. The condition $LG.x = LG.(x + w)$, $\forall w \in W$ does not hold in general. For example, let $\sigma = x^2 + xy^2 \in J^4(2, 1)$, W the space of homogeneous polynomials of degree 4, and $\mathscr{R}^{(4)}$. Then $\sigma + w = x^2 + xy^2 + ty^4$, and we can see that the 4-jets when $t = 0$ or $t = 1/4$ are not equivalent in $j^4(2, 1)$.

Proof. (of Theorem 3.3.1)
(a) We want to prove that

$$x + \{LG.x \cap W\} \subset G.x$$

By hypothesis, $\forall w \in LG.x \cap W$, the condition $LG(x + w) = LG.x$ holds. As the condition $LG.x \cap W \subset G.x$ clearly holds, we can apply Corollary 3.3.2 to prove

$$x + \{LG.x \cap W\} \subset G.x.$$

To prove (b) let T be a vector subspace such that $LG.x + T \supset W$. Then,

$$\cup_{t\in T} G.(x_0 + t) = \cup_{t\in T} \{x_0 + t + \{LG.(x_0 + t) \cap W\}\} \supset x_0 + W$$

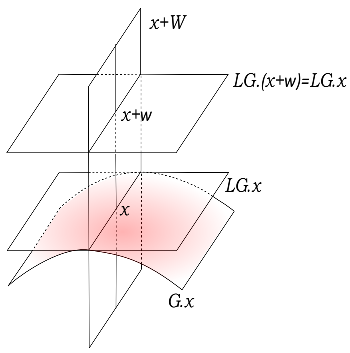


Figure:

□

The next result is the complete transversal theorem for the group

$$\mathscr{R}_1 = \{h \in \mathscr{R} \mid j^1 h(0) = Id_{\mathbb{K}^n}\}.$$

Theorem 3.3.4. (Complete transversal for the group $\mathscr{R}_1$)

Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a polynomial of degree k and let $\{g_1, \dots, g_n\}$ a collection of homogeneous polynomials of degree $k+1$ with the property that

$$\mathfrak{m}_n^2 Jf + \mathbb{K}\{g_1, \dots, g_r\} + \mathfrak{m}_n^{k+2} \supset \mathfrak{m}_n^{k+1}.$$

Then, for all $g \in J^{k+1}(n,1)$ such that $j^k g(0) = j^k f(0)$ is $\mathscr{R}_1^{k+1}$-equivalent to

$$f(x) + \Sigma_{i=1}^r u_i g_i(x), \; u_i \in \mathbb{K}.$$

Proof. Let $A = J^{k+1}(n,1)$ and $W = \frac{\mathfrak{m}_n^{k+1} + \mathfrak{m}_n^{k+2}}{\mathfrak{m}_n^{k+2}}$. If $\sigma = j^k f$, then $\sigma + W = \sigma + H^{k+1}(n,1)$.

Let g be any germ such that $g \in \sigma + W$, that is $g - f \in \mathfrak{m}_n^{k+1}$. Then it follows that

$$\mathfrak{m}_n^2 Jg + \mathfrak{m}_n^{k+2} = \mathfrak{m}_n^2 Jf + \mathfrak{m}_n^{k+2},$$

and we can apply Theorem 3.3.4 to get that if $T = \{g_1, \dots, \gamma_r\}$.

□

Exercises

1. Show that $f(x_1, x_2, x_3, x_4) = x_1^3 + x_2^3 + x_3^3 + x_4^3$ is 4-determined with respect to $\mathscr{R}$-equivalence.

2. Use the complete transversal method to determine all 4-jets whose 3-jet is equivalent to f given in exercise 1.

3. Show that f in exercise 1. is not 3-determined and compute $\mu(f)$.

4. Use the complete transversal to determine all $\mathscr{R}^3$-orbits in $J^3(3,1)$ of 3-jets whose 2-jet is $j^2 f(x,y,z) = xy$.

3.4 First steps of ADE-classification

In chapter 6 we will prove the theorem of Arnold (see Corollary 6.2.5), which gives the classification of $\mathscr{R}$-simple singularities. This classification is known as ADE-classification. These singularities are $\mathscr{R}$-equivalent to A_k, D_k, E_6, E_7 and E_8. In this section, we classify the singularities A_k and D_k, and find the degree of determinacy with respect to $\mathscr{R}$-equivalence of all relevant normal forms for a function germ

$$f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0).$$

We start with the A_k singularities of functions of one variable.

Proposition 3.4.1. *Let $g\colon (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_1^3$ and $\mu(g) = k \geq 2$. Then g is $\mathscr{R}$-equivalent to the A_k-singularity $\pm x^{k+1}$.*

Proof. Since $\mu(g) = k$ we have $J(g) = (x^k)$ and hence $g(x) = a(x)x^{k+1}$ for some $a \in \mathscr{O}_1$ such that $a(0) \neq 0$. Assume that either $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and k is even or $\mathbb{K} = \mathbb{R}$, k is odd and $a(0) > 0$. We take the diffeomorphism $\phi(x) = a(x)^{1/(k+1)}x$, which gives the $\mathscr{R}$-equivalence of g with x^{k+1}. When $\mathbb{K} = \mathbb{R}$, k is even and $a(0) < 0$ we must take $\phi(x) = (-a(x))^{1/(k+1)}x$, which makes g $\mathscr{R}$-equivalent to $-x^{k+1}$.

□

Example 3.4.2. An $A_k,\ k \geq 1$ singularity at the origin in $\mathbb{K}^2$ is $\mathscr{R}$-equivalent to

$$\begin{aligned} f_k : (\mathbb{K}^2, 0) &\to (\mathbb{K}, 0) \\ (x, y) &\mapsto f_k(x, y) = \pm x^2 \pm y^{k+1}. \end{aligned}$$

We can show easily that f_k is $k+1$-determined with respect to $\mathscr{R}$-equivalence, and $\mu(f) = k$.

Proposition 3.4.3. *Let $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_2^3$, $\mu(g) = k \geq 4$ and j^3g is either non-degenerate (hyperbolic or elliptic in the real case) or parabolic. Then g is $\mathscr{R}$-equivalent to the $D_k^{\pm}$-singularity*

$$x^2y \pm y^{k-1}.$$

Proof. Suppose first that j^3g is non-degenerate, so it is $\mathscr{R}$-equivalent to $\sigma = x^2y \pm y^3$ by Proposition 1.2.8. We have $\mathfrak{m}_2 J(\sigma) = \mathfrak{m}_2^3$ and thus, σ is 3-determined, by the Finite Determinacy Theorem. Therefore, g is $\mathscr{R}$-equivalent to $D_4^{\pm}$.

When j^3g is parabolic, it is $\mathscr{R}$-equivalent to x^2y, again by Proposition 1.2.8. Here we use the complete transversal method. We consider $\sigma = x^2y$ as an ℓ-jet in $J^\ell(2,1)$, with $\ell \geq 3$. The $\mathscr{R}_1^{(\ell+1)}$-orbit of σ is

$$\frac{\mathfrak{m}_2^2 J(\sigma) + \mathfrak{m}_2^{\ell+1}}{\mathfrak{m}_2^{\ell+1}},$$

which contains all monomials of degree $\ell+1$ except $y^{\ell+1}$. Hence a complete transversal for σ is the subspace spanned by $y^{\ell+1}$. By the complete transversal theorem, $j^{\ell+1}g$ is $\mathscr{R}_1^{\ell+1}$-equivalent to $x^2y + ay^{\ell+1}$, for some $a \in \mathbb{K}$. If $a \neq 0$, we get $\sigma' = x^2y \pm y^{\ell+1}$. A simple computation shows that σ' is $(\ell+1)$-determined, hence f is $\mathscr{R}$-equivalent to $D_k^{\pm}$, with $k = \ell+2$. If $a = 0$, we get again x^2y and proceed to the next jet space. Since f is $\mathscr{R}$-finite, this process must stop at some ℓ and hence this gives all possible orbits when j^3g is parabolic.

□

Remark 3.4.4. As it happened with the A_k-singularities, the two germs D_k^+ and D_k^- are $\mathscr{R}$-equivalent when $\mathbb{K} = \mathbb{C}$ or when $\mathbb{K} = \mathbb{R}$ and k is even and are denoted by D_k. However, D_k^+ and D_k^- are not $\mathscr{R}$-equivalent when $\mathbb{K} = \mathbb{R}$ and k is odd.

To conclude the ADE-classification we still need to consider the case in which j^3g is a symbolic cubic form. These will lead to the E_6, E_7, E_8 singularities, to be discussed in chapter 6.

We discuss in the next example the degree of determinacy of these singularities.

Example 3.4.5. The singularities E_6, E_7, E_8 have the normal forms:

$$\begin{aligned} E_6 &: f(x,y) = x^3 \pm y^4,\ \mu(f) = 6 \\ E_7 &: f(x,y) = x^3 + xy^3,\ \mu(f) = 7 \\ E_8 &: f(x,y) = x^3 + y^5,\ \mu(f) = 8, \end{aligned}$$

and they are respectively, 4, 4 and 5 determined with respect to $\mathscr{R}$-equivalence. The calculations of the Milnor number and the degree of determinacy for the normal forms of E_6 and E_8 are not hard, and we leave them as exercises.

We show that the singularity E_7 is 4-determined and its Milnor number is 7. Using the Complete Transversal Method, it follows that any 5-jet whose 4-jet is $x^3 + xy^3$ is $\mathscr{R}^{(5)}$-equivalent to $f_t(x,y) = x^3 + xy^3 + ty^5$, for some $t \in \mathbb{K}$.

We now prove that

$$\mathfrak{m}_2 J(f_t) + \mathfrak{m}_2^6 = \left\langle 3x^3 + xy^3, 3x^2y + y^4, 3x^2y^2 + 5txy^4, 3xy^3 + 5ty^5 \right\rangle + \mathfrak{m}_2^6 \subset \mathfrak{m}_2^5.$$

The result follows by Mather's lemma and the finite determinacy theorem.

One can check that

$$\frac{\mathscr{O}_2}{J(f)} \backsimeq \mathbb{K}\left\{1, x, y, xy, y^2, y^3, y^4\right\},$$

hence $\mu(x^3 + xy^3) = 7$.

Part III

Lecture 3

Chapter 4

Deformations. Versality.

Since the beginning of singularity theory it has been clear that in order to understand a singularity you have to understand what happens when you deform it into less degenerate types of singularities. For instance, when looking at a bent wire from the tangent direction at a point of the wire where it has 0 torsion you see a cusp. In order to understand what is going on with the wire you must move your head slightly to the left and to the right, from one side you will see a regular piece of wire and from the other side you will see a kind of loop. We have deformed the cusp and by seeing what happens near the cusp we have understood how this singularity appears. In a certain sense we need a family (a 1-paramater family in this case) of views in order to grasp the full nature of the singularity. In this part of the lecture we will give the definition of unfolding and deformation of a smooth function germ $f\colon (\mathbb{K}^n,0)\to(\mathbb{K},0)$. When working with map germs $f\colon (\mathbb{K}^n,0)\to(\mathbb{K}^p,0)$, $p>1$ it is natural to consider $\mathscr{A}$-equivalence (i.e. smooth changes of coordinates in source and target) and the notion of unfolding is crucial. However, since our focus is on function germs, it is enough to consider deformations and $\mathscr{R}$-equivalence. We will explain the concept of versality, one of the central ideas in the theory of singularities of map germs, and show several characterizations. We will follow the approach in [16] and [20].

4.1 Basic definitions and examples

Let $f\colon (\mathbb{K}^n,0)\to(\mathbb{K},0)$ be a smooth function germ, where smooth means C^∞ when $\mathbb{K}=\mathbb{R}$ or holomorphic when $\mathbb{K}=\mathbb{C}$.

Definition 4.1.1. i) A *d-parameter unfolding* of f is a map-germ

$$F'\colon (\mathbb{K}^n\times\mathbb{K}^d,0)\to(\mathbb{K}\times\mathbb{K}^d,0)$$

of the form $F'(x,u)=(F(x,u),u)$ such that $F(x,0)=f(x)$. If we denote $F(x,u)$ by $f_u(x)$, the above condition becomes $f_0=f$. The map-germ

$$F\colon (\mathbb{K}^n\times\mathbb{K}^d,0)\to(\mathbb{K},0)$$

is a *d-parameter deformation* of f.

ii) Two deformations F,G of f are *equivalent* if there exists a germ of diffeomorphism

$$\Phi\colon (\mathbb{K}^n\times\mathbb{K}^d,0)\to(\mathbb{K}^n\times\mathbb{K}^d,0)$$

of the form $\Phi(x,u) = (\varphi_u(x), u)$ where $\varphi(x,0) = x$ (i.e. Φ is a deformation of the identity in $\mathbb{K}^n$), such that $G = F \circ \Phi$.

iii) A deformation F is called *trivial* if it is equivalent to the constant deformation $\tilde{F}(x,u) = f(x)$.

iv) A map-germ is called *stable* if any deformation of it is trivial.

Example 4.1.2. Consider the germ $f\colon (\mathbb{K},0) \to (\mathbb{K},0)$ given by $f(x) = x$ and the deformations $F(x,u) = x$ and $G(x,u) = x + ux^2$. Taking the diffeomorphism $\Phi(x,u) = (x+ux^2, u)$ we get $F \circ \Phi = G$ and so F and G are equivalent. Furthermore, since F is the constant deformation, this means that G is trivial.

In fact, given any deformation $H(x,u)$, since $H(x,0) = x$, by considering the diffeomorphism $\Phi(x,u) = (H(x,u), u)$ we see that H is trivial. We have shown that $f(x) = x$ is stable.

Remark 4.1.3. It can be seen that a function germ $f\colon (\mathbb{K}^n,0) \to (\mathbb{K},0)$ is $\mathscr{R}_e$-stable if and only if $\dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f)} = 0$, and, thus, if and only if df_0 is surjective, i.e. the function is regular. This is completely different for map-germs and $\mathscr{A}$-stability.

Deformations allow us to see what happens around a singularity, but in order to understand the singularity completely we want to know what are *all* the possible phenomena that appear around it. The idea of a versal deformation is that it captures all the possible less degenerate singularities into which a certain singularity can be deformed.

Definition 4.1.4. i) Let $F : (\mathbb{K}^n \times \mathbb{K}^d, 0) \to (\mathbb{K},0)$ be a deformation of a map-germ f and let $h : (\mathbb{K}^a,0) \to (\mathbb{K}^d,0)$ be a map-germ. The *pull-back* of F by h is the deformation

$$h^*F : (\mathbb{K}^n \times \mathbb{K}^a, 0) \to (\mathbb{K},0)$$

given by

$$h^*F(x,v) = F(x,h(v)).$$

The map-germ h is called the *base change map*.

ii) A d-parameter deformation F of f is *versal* if for any a-parameter deformation G there is a map-germ $h : (\mathbb{K}^a,0) \to (\mathbb{K}^d,0)$ such that G is equivalent to h^*F. It is called *miniversal* if there is no versal deformation with less than d parameters.

iii) Two d-parameter deformations F and G of f are *isomorphic* if there exists a diffeomorphism $h : (\mathbb{K}^d,0) \to (\mathbb{K}^d,0)$ such that G is equivalent to h^*F.

Example 4.1.5. Consider the function $f(x,y) = x^2 + y^3$ and the deformations $F(x,y,u_1,u_2) = x^2 + y^3 + u_1 y + u_2$ and $G(x,y,v) = x^2 + y^3 + 3vy^2$. Given the map-germ $h(v) = (-3v^2, 2v^3)$, we get $h^*F(x,y,v) = x^2 + y^3 - 3v^2 y + 2v^3$. Using the diffeomorphism $\Phi(x,y,v) = (x, y+v, v)$ we see that $G = h^*F \circ \Phi$, i.e. G is equivalent to h^*F. We will see in the next section that F is, in fact, versal.

4.2 Characterizations of versality

The definition of versality is useless in order to prove when a certain deformation F is versal, we cannot find a map-germ h for any other possible unfolding G, so we need certain characterizations in order to find or prove when an unfolding is versal.

Remember that we have defined $T\mathscr{R}_e(f)$ as the Jacobian ideal $J(f)$ and that the $\mathscr{R}_e$-codimension of f is $\dim_{\mathbb{K}} \frac{\theta(f)}{T\mathscr{R}_e(f)}$. Let $F(x,u) = f_u(x)$ be an unfolding of a function germ f. We denote $\frac{\partial f_u}{\partial u_i}|_{u=0}$ by $\dot{F}_i$. The following is an infinitesimal criterion for versality due to Martinet ([15]). We will prove only the necessity of the criterion, the proof of sufficiency is much longer and exceeds the reach of this lecture notes. The proof is based on a fundamental geometrical lemma of the theory of deformations concerning the existence of a certain liftable vector field. A detailed account can be found in Chapters IV and XI in [16] or Theorem 5.1 in [20] for the $\mathscr{A}$-equivalence version.

Theorem 4.2.1. *The d-parameter deformation F of f is versal if and only if*

$$J(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\} = O_n,$$

i.e. the functions $\dot{F}_1, \ldots, \dot{F}_d$ generate $\frac{O_n}{J(f)}$ as a $\mathbb{K}$-vector space.

Proof. Let $g \in O_n$ and consider a 1-parameter deformation $G(x,v) = f(x) + vg(x)$ of f. Notice that $\dot{G} = g$. Since $F(x,u) = f_u(x)$ is versal, there exists $h : (\mathbb{K}, 0) \to (\mathbb{K}^d, 0)$ such that $G(x,v)$ is equivalent to $h^*F(x,v) = f_{h(v)}(x)$. Applying the chain rule we get

$$\dot{h^*F} = \frac{d(f_{h(v)})}{dv}|_{v=0} = \sum_{i=1}^{d} h_i'(0)\frac{\partial f_u}{\partial u_i}|_{u=0} = \sum_{i=1}^{d} h_i'(0)\dot{F}_i \in Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}.$$

Since G is equivalent to h^*F, there exists a diffeomorphism $\Phi(x,v) = (\varphi_v(x), v)$ such that $G = h^*F \circ \Phi$, or, alternatively, $G = f_{h(v)} \circ \varphi_v$. Applying the chain rule again and taking into account that $f_{h(0)} = f$ and that φ_0 is the identity we get

$$g = \frac{d}{dv}(f_{h(v)} \circ \varphi_v)|_{v=0} = \sum_{i=1}^{n} \frac{\partial f_{h(v)}}{\partial x_i} \circ \varphi_v|_{v=0} \frac{d(\varphi_v)_i}{dv}|_{v=0} + \frac{\partial f_{h(v)}}{dv}|_{v=0} = df \circ \frac{d\varphi_v}{dv}|_{v=0} + \frac{\partial f_{h(v)}}{dv}|_{v=0}.$$

The first term in the right-hand side of the equation belongs to $J(f)$ and the second term, by the above argument belongs to $Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}$, so we get $g \in J(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}$. □

Remark 4.2.2. The infinitesimal criterion for function germs can be expressed as

$$T\mathscr{R}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\} = \theta(f),$$

which is similar to the criterion for $\mathscr{A}_e$-versality but changing the $\mathscr{A}_e$-tangent space by the $\mathscr{R}_e$-tangent space.

Example 4.2.3. i) The deformation $F(x,y,u_1,u_2) = x^2 + y^3 + u_1 y + u_2$ of Example 4.1.5 is versal since $\frac{\mathscr{O}_n}{J(f)}$ is generated by y and 1. This explains why $G(x,y,v) = x^2 + y^3 + 3vy^2$ is equivalent to h^*F for some h. In fact, any other deformation H will be equivalent to a pull-back of F.

ii) Consider $f(x) = x^4$, $\frac{\mathscr{O}_n}{J(f)}$ is generated by $\{1, x, x^2\}$ so a versal deformation is $F(x,u_1,u_2,u_3) = x^4 + u_1 x^2 + u_2 x + u_3$. The parameter u_3 is just a translation. If you consider the plane u_1, u_2, for every point in the plane you get a different function. It is interesting to see how this function varies and what singularities appear. For instance, along the curve $(-6s^2, 8s^3)$, the function has an inflection point at the origin. On one side of this curve the function has two local minima and one local maximum, on the other side there is just one local minimum. This is called a *bifurcation diagram*, we refer the reader to [20] for more details on this set.

If we consider the unfolding $G(x, u_1) = x^4 + u_1 x^2$, as u_1 varies we will appreciate changes in the function, namely it has 3 critical points when $u_1 < 0$ and 1 critical point otherwise. However, this deformation is not versal, in particular it does not show how in any neighbourhood of the function f there are functions with inflection points.

On the other hand, the unfolding $H(x, y, u_1, u_2, u_3, u_4) = x^4 + u_1 x^2 + u_2 x + u_3 + u_4 x^3$ is also versal but it is not miniversal, since F has less parameters than H. In fact, H can be seen as a trivial deformation of F.

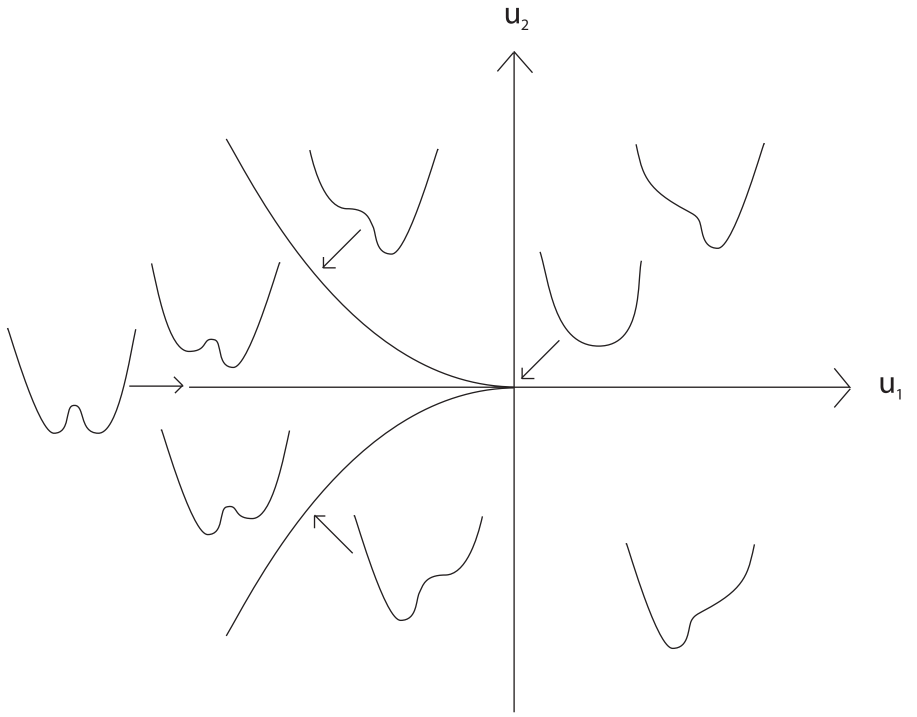


Figure 4.1: Different functions for different values of u_1 and u_2 represented in the $\{u_1, u_2\}$-plane.

Corollary 4.2.4. *A function germ f admits a versal deformation if and only if its $\mathscr{R}_e$-codimension is finite. Moreover, the $\mathscr{R}_e$-codimension is equal to the number of parameters in a miniversal deformation.*

Proof. Given a versal d-parameter deformation F, by the versality criterion, $\dot{F}_1, \ldots, \dot{F}_d$ generate $\frac{\mathscr{O}_n}{J(f)}$ as a $\mathbb{K}$-vector space, so $\mathscr{R}_e\text{-cod}(f) = \dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f)} \leq d$. Converserly, if $\mathscr{R}_e\text{-cod}(f) = d$, there exist $g_1, \ldots, g_d \in O_n$ whose classes generate $\frac{\mathscr{O}_n}{J(f)}$ over $\mathbb{K}$, so $F(x, u) = f(x) + \sum_{i=1}^{d} u_i g_i(x)$ is a miniversal deformation of f. □

Corollary 4.2.5. *Any two versal deformations with the same number of parameters are isomorphic.*

Proof. Suppose first that F, G are two d-parameter miniversal deformations. Since F is versal, there exists $h : (\mathbb{K}^d, 0) \to (\mathbb{K}^d, 0)$ such that G is equivalent to h^*F. Since G is versal, h^*F is versal too. Applying the chain rule to $h^*F = f_h$ we get

$$(\dot{h^*F})_i = \sum_{i=1}^{d} \frac{\partial h_j}{\partial u_i}(0)\dot{F}_j$$

for $i = 1, \dots, d$. Since both F and h^*F are miniversal, $\{\dot{F}_1, \dots, \dot{F}_d\}$ and $\{(\dot{h^*F})_1, \dots, (\dot{h^*F})_d\}$ are bases of $\frac{\mathcal{O}_n}{J(f)}$, and so $(\frac{\partial h_j}{\partial u_i}(0))$ is an invertible matrix. This means that h is a diffeomorphism and so G and F are isomorphic.

Now suppose F, G are versal m-parameter deformations with $m > d$. We have $\dim_{\mathbb{K}} Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_m\} = d$, so there are $m-d$ linear combinations of the $\dot{F}_i$ which give 0. This means that there exists a linear change of parameters $h_1 : (\mathbb{K}^m, 0) \to (\mathbb{K}^m, 0)$ such that h_1^*F verifies that there exists $m-d$ partials $(\dot{h_1^*F})_i$ which are 0, i.e. h_1^*F is a constant deformation of a miniversal deformation. Similarly, there exists h_2 such that h_2^*G is a constant deformation of a miniversal deformation. Since h_1 and h_2 are diffeomorphisms, F and G are isomorphic. □

We end this section by showing how the versality of a deformation can be characterized by the transversality of the family of jet extension maps to the $\mathscr{R}$-orbit of f in the jet space. This result is true for $\mathscr{A}_e$-versal unfoldings and $\mathscr{A}$-orbits. We prove it here for $\mathscr{R}$-equivalence.

Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a function germ and F a d-parameter deformation. Consider X and U neighbourhoods of 0 in $\mathbb{K}^n$ and $\mathbb{K}$ respectively such that $F : X \times U \to \mathbb{K}$ is a representative of the deformation. Consider the *relative jet extension map* or *partial jet extension map*

$$j^k F/U : X \times U \to J^k(X, \mathbb{K})$$

given by $(x, u) \mapsto j^k f_u(x)$.

We denote by $\mathscr{R}^{(k)}$ de k-jets of diffeomorphisms in the group $\mathscr{R}$. Given an $\mathscr{R}^{(k)}$-invariant subset $W \subset J^k(n, 1)$, we can consider the induced subbundle $W(X, \mathbb{K}) \subset J^k(X, \mathbb{K})$ with base space $X \times \mathbb{K}$ and fibre W.

Theorem 4.2.6. *Suppose that $\mathfrak{m}_n^{k+1}\mathcal{O}_n \subseteq T\mathscr{R}(f)$. The deformation F of f is versal if and only if $j^k F/U$ is transverse to $W(X, \mathbb{K})$, where W is the $\mathscr{R}^{(k)}$-orbit of $j^k f$ in $J^k(n, 1)$.*

Proof. At the point $z = (x, 0)$ the image of the differential $d_z(j^k F/U)$ is generated by the k-jets of $\frac{\partial f_u}{\partial x_i}|_z = \frac{\partial f}{\partial x_i}|_x = df(\frac{\partial}{\partial x_i})$, $i = 1, \dots, n$, and $\frac{\partial f_u}{\partial u_i}|_{u=0} = \dot{F}_i$, $i = 1, \dots, d$. Therefore, $j^k F/U$ is transverse to $W(X, \mathbb{K})$ at z if and only if

$$T\mathscr{R}(f) + Sp_{\mathbb{K}}\{df(\frac{\partial}{\partial x_1}), \dots, df(\frac{\partial}{\partial x_n})\} + Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_d\} + \mathfrak{m}_n^{k+1}\theta(f) = \theta(f).$$

Since $T\mathscr{R}(f) = \mathfrak{m}_n J(f)$, then $T\mathscr{R}(f) + Sp_{\mathbb{K}}\{df(\frac{\partial}{\partial x_1}), \dots, df(\frac{\partial}{\partial x_n})\} = T\mathscr{R}_e(f)$. On the other hand, since $\mathfrak{m}_n^{k+1}\mathcal{O}_n \subseteq T\mathscr{R}(f)$ we get

$$T\mathscr{R}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_d\} = \theta(f),$$

which is equivalent to versality of F by the versality criterion. □

Exercises

1. Let $f(x, y) = x^3 + y^2 x$.

   i) Find a miniversal deformation.

   ii) Show that $F(x, y, u) = x^3 + y^2 x + 2ux^2 y$ is a trivial deformation of f.

2. Show that if two deformations are equivalent, then they are isomorphic.

3. Show that if h is a diffeomorphism in the parameter space, then F and $h^* F$ are isomorphic.

4. Show that f is stable if and only if the $\mathscr{R}_e$-codimension is 0.

5. Show that if f is stable, then all deformations are versal.

6. Show that any versal deformation is isomorphic to the constant deformation of a miniversal deformation.

7. Show that any versal deformation of a function germ is stable as a function germ.

8. Show that if the $\mathscr{R}_e$-codimension of f is 1, then any stable deformation is versal.

Chapter 5

The Splitting Lemma

As an application of the complete transversal's method we will give a result which will be needed in the next lecture in order to classify the ADE-singularities. This result is due to René Thom.

Lemma 5.0.1. *Let* $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ *be such that its* $\mathscr{R}_e$*-codimension is finite. If the Hessian matrix of* f *has corank c, then* f *is* $\mathscr{R}$*-equivalent to a function of the form*

$$g(x_1, \ldots, x_c) \pm x_{c+1}^2 \pm \ldots \pm x_n^2,$$

for some $g \in \mathfrak{m}_c^3$ *such that the* $\mathscr{R}_e$*-codimension of g coincides with that of* f.

Proof. Since the Hessian of f has corank c, $j^2 f$ is equivalent, up to linear changes of coordinates, to $\pm x_{c+1}^2 \pm \ldots \pm x_n^2$. We will prove by induction that for any $k \geq 2$ $j^k f$ is $\mathscr{R}^{(k)}$-equivalent to a k-jet of the form

$$\tau_k = \sigma_k \pm x_{c+1}^2 \pm \ldots \pm x_n^2,$$

for some $\sigma_k \in J^k(c, 1) \cap \mathfrak{m}_c^3$. For the case $k = 2$ take $\sigma_2 = 0$. Suppose it is true for k and lets prove it for $k + 1$. Notice that $T\mathscr{R}^{(k+1)}(\tau_k)$ is generated by $\frac{\partial \sigma_k}{\partial x_1}, \ldots, \frac{\partial \sigma_k}{\partial x_n}$ and $x_{c+1}, \ldots, x_n$. Therefore, a complete transversal for τ_k in $J^{k+1}(n, 1)$ is given by the homogeneous polynomials of degree $k + 1$ in the variables $x_1, \ldots, x_c$, i.e. $H_c^{k+1} \subset J^{k+1}(c, 1)$. By the complete transversal result, $j^{k+1} f$ will be $\mathscr{R}^{(k+1)}$-equivalent to

$$\tau_{k+1} = \sigma_k + h \pm x_{c+1}^2 \pm \ldots \pm x_n^2,$$

for $h \in H_c^{k+1}$. Now, since f has finite codimension, it is finitely determined and so f is $\mathscr{R}$-equivalent to $j^k f$ for some k big enough, and so f will be equivalent to the desired form.

On the other hand

$$\dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f)} = \dim_{\mathbb{K}} \frac{\mathscr{O}_n}{(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_c}, x_{c+1}, \ldots, x_n)} = \dim_{\mathbb{K}} \frac{\mathscr{O}_c}{Jg},$$

and so the codimensions of f and g are equal. □

Remark 5.0.2. i) When $\mathbb{K} = \mathbb{C}$ all the signs of the quadratic part can be taken to be $+1$. When $\mathbb{K} = \mathbb{R}$ the signs are defined by the signature of the Hessian matrix (see the applications of Mather's Lemma in Lecture 1).

ii) The corank 0 case corresponds to the Morse Lemma.

Exercises

1. Show that when $\mathbb{K} = \mathbb{C}$ all the signs of the quadratic part can be taken to be $+1$ and when $\mathbb{K} = \mathbb{R}$ the signs are defined by the signature of the Hessian matrix.

2. Suppose f and f' have the same corank (of the Hessian). Show that f and f' are $\mathscr{R}$-equivalent if and only if the corresponding g and g' are $\mathscr{R}$-equivalent.

3. Show that the versal deformation of f can be obtained from the versal deformation of g.

Part IV

Lecture 4

Chapter 6

The ADE-classification

In this chapter we will focus on classification of smooth function germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ up to $\mathscr{R}$-equivalence. We recall that smooth means C^∞ when $\mathbb{K} = \mathbb{R}$ or holomorphic when $\mathbb{K} = \mathbb{C}$. The first classification result in this context was obtained by René Thom for singularities up to $\mathscr{R}_e$-codimension 5 when he was working on his famous *Catastrophe Theory* [26]. He found that there are seven classes of singularities which are known as the *seven elementary catastrophes* of Thom. The next important result was obtained by Siersma in [24], who extended the classification for singularities up to $\mathscr{R}_e$-codimension 9.

Here we will give an important theorem by Arnold [1], which is known as the ADE-classification. It says that any simple function germ is $\mathscr{R}$-equivalent to A_k, D_k or one of the three exceptional germs E_6, E_7 or E_8. The ADE terminology is due to the connection with the classification of simple Lie algebras (see [1] for details). Other common names for the ADE-singularities are Du Val singularities, Kleinian singularities or rational double points (see for instance [8, 22]).

We start with the definition of simple singularity. Then we will use the complete transversal method to show how to obtain the ADE-singularities and that all of them are simple. The second section will be devoted to prove that they are the only simple singularities. Some of the statements and proofs of this chapter have been taken from the book by De Jong and Pfister [6].

6.1 Simple singularities

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a smooth function germ. Unless otherwise stated, all deformations $F(x, u) = f_u(x)$ of f will be origin preserving, that is, $f_u(0) = 0$, for all u close to 0, so we can consider the germ $f_u\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$.

Definition 6.1.1. Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a smooth function germ. We say f is *simple* if there exist a finite number of $\mathscr{R}$-classes such that for any deformation $F(x, u) = f_u(x)$, the germs of f_u at the origin all lie in one of these $\mathscr{R}$-classes.

Example 6.1.2. Any regular (A_0) germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ is simple. In fact, we have $df_0 \neq 0$, where df_0 is the differential of f at 0. By continuity of the partial derivatives we have $d(f_u)_0 \neq$ for all u close to 0. This implies that the germ f_u is regular. Hence f is simple.

An analogous argument shows that any Morse singularity (A_1) is also simple. We leave the proof as an exercise (Exercise 1).

Example 6.1.3. Consider the function $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ given by $f(x, y) = xy(x-y)(x+y)$ and the 1-parameter deformation F defined by

$$f_t(x, y) = (x + ty)y(x - y)(x + y).$$

If f_t is $\mathscr{R}$-equivalent to $f_{t'}$ at the origin, then there exists a diffeomorphism $\varphi : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ such that $\varphi(f_t^{-1}(0)) = f_{t'}^{-1}(0)$. Since f_t and $f_{t'}$ are both homogeneous of degree 4, this implies that $d_0\varphi(f_t^{-1}(0)) = f_{t'}^{-1}(0)$. Now $f_t^{-1}(0)$ and $f_{t'}^{-1}(0)$ are both a union of 4 lines in $\mathbb{K}^2$ passing through the origin and $d_0\varphi$ is a linear isomorphism which must preserve the cross-ratio between both sets (see fig. 6.1). Hence $t = t'$, for $t, t' \in \mathbb{K}$ close to the origin.

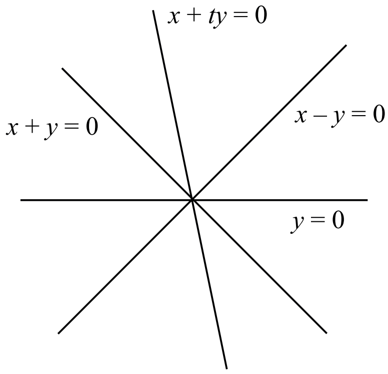


Figure 6.1: The four lines defined by the zeros of f_t

Given a smooth function $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, its $\mathscr{R}_e$-codimension is also called the *Milnor number* of f and it is denoted by $\mu(f)$, that is,

$$\mu(f) := \mathscr{R}_e - \operatorname{codim}(f) = \dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f)}.$$

We will see in Section 7.3 its relationship between with the Milnor fibration in the complex case which makes clear this terminology. When $\mu(f) < \infty$, it is enough to check the simplicity of f in its $\mathscr{R}$-versal deformation. The $\mathscr{R}$-versal deformation is the origin preserving version of the versal deformation (see Definition 4.1.4), but using only origin preserving deformations. Moreover, in this case a deformation $F(x, u) = f_u(x)$ is $\mathscr{R}$-versal if and only if the classes of $\dot{F}_1, \ldots, \dot{F}_d$ generate $\mathfrak{m}_n/\mathfrak{m}_n J(f)$ as a $\mathbb{K}$-vector space (see Theorem 4.2.1).

Lemma 6.1.4. *Assume that $\mu(f) < \infty$ and let $F(x, u) = f_u(x)$ be its $\mathscr{R}$-versal deformation. Then f is simple if and only if there exist a finite number of $\mathscr{R}$-classes such that the germs of f_u at the origin all lie in one of these $\mathscr{R}$-classes.*

Proof. The only if part is obvious. To see the if part, let $G(x, v) = g_v(x)$ be any deformation of f. By versality, there exist a diffeomorphism $\Phi : (\mathbb{K}^n \times \mathbb{K}^s, 0) \to (\mathbb{K}^n \times \mathbb{K}^s, 0)$ which is an unfolding of the identity and a base change map $h : (\mathbb{K}^s, 0) \to (\mathbb{K}^r, 0)$ such that $G \circ \Phi = h^*F$. In particular, $g_v \circ \phi_v = f_{h(v)}$, so the germ of g_v at the origin lies in one of the $\mathscr{R}$-classes of F. □

We recall that if $0 < \mu(f) < \infty$, then the corank of f is defined as the corank of its Hessian matrix. By the Splitting Lemma, if f has corank c, then up to $\mathscr{R}$-equivalence, f can be written as

$$f(x) = g(x_1, \ldots, x_c) \pm x_{c+1}^2 \pm \cdots \pm x_n^2, \tag{6.1}$$

for some germ $g \in \mathfrak{m}_c^3$ such that $\mu(g) = \mu(f)$. In the complex case $\mathbb{K} = \mathbb{C}$, all $\pm$ in the the quadratic part of (6.1) can be converted into $+$. In the real case $\mathbb{K} = \mathbb{R}$, the $\pm$ depend on the signature of the Hessian matrix of f at 0.

Moreover, if two germs f_1 and f_2 with the same same corank and signature (in the real case) are written as in (6.1) for some germs g_1 and g_2, then f_1 and f_2 are $\mathscr{R}$-equivalent if and only if so are g_1 and g_2. So, in order to obtain a classification is enough to consider germs $g \in \mathfrak{m}_c^3$ with $\mu(g) < \infty$.

Finally, the versal deformation of f given by (6.1) can be obtained as

$$f_u(x) = g_u(x_1, \dots, x_c) \pm x_{c+1}^2 \pm \cdots \pm x_n^2,$$

where g_u is the versal deformation of g. If follows that f is simple if and only if so is g.

The case of corank 0 is equivalent to that $\mu(f) = 1$ and corresponds to a Morse singularity (A_1). The first result in our ADE-classification is when f has corank 1, so we look at germs in one variable.

Proposition 6.1.5. *Let $g\colon (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_1^3$ and $\mu(g) = k \geq 2$. Then g is $\mathscr{R}$-equivalent to the A_k-singularity $\pm x^{k+1}$. Moreover, g is simple.*

Proof. The proof that g is $\mathscr{R}$-equivalent to the A_k-singularity $\pm x^{k+1}$ was given in Proposition 6.1.5. Let g_u be the versal deformation of g. For all u close to 0, $\mu(g_u) = \ell \leq k$ by upper semicontinuity of the Milnor number (see Corollary 7.1.3). It follows that g_u is $\mathscr{R}$-equivalent to A_ℓ. Therefore, g is simple. □

The next step is when f has corank two, so we look now at functions $g \in \mathfrak{m}_2^3$ in two variables. Observe that we have $\mu(g) \geq 4$ (see Exercise 2). Furthermore, its 3-jet $j^3 g$ is a cubic form (i.e., a homogeneous polynomial of degree 3).

Proposition 6.1.6. *Let $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_2^3$, $\mu(g) = k \geq 4$ and the cubic form $j^3 g$ is either non-degenerate (hyperbolic or elliptic in the real case) or parabolic. Then g is $\mathscr{R}$-equivalent to the $D_k^\pm$-singularity*

$$x^2 y \pm y^{k-1}.$$

Moreover, g is simple.

Proof. The proof that g is $\mathscr{R}$-equivalent to the $D_k^\pm$-singularity $x^2 y \pm y^{k-1}$ was given in Proposition 3.4.3. Let g_u be the versal deformation of g. For all u close to 0, g_u is either regular, Morse or has corank ≤ 2 and $\mu(g_u) = \ell \leq k$ by Lemma 7.1.3. Moreover, since $j^3 g$ is non-degenerate or parabolic, then $j^3 g_u$ is also non-degenerate or parabolic when g_u has corank 2. It follows that g_u is either regular, Morse or $\mathscr{R}$-equivalent to A_ℓ or $D_\ell^\pm$. Hence, g is simple. □

To finish the ADE-classification we consider the case that $j^3 g$ is a symbolic cubic form. We left as an exercise to show that this implies $\mu(g) \geq 6$. We also put the additional restriction that $\mu(g) \leq 8$.

Proposition 6.1.7. *Let* $g\colon (\mathbb{K}^2,0)\to(\mathbb{K},0)$ *be smooth such that* $g\in\mathfrak{m}_2^3$, $\mu(g)=k$ *with* $k=6,7$ *or* 8 *and* j^3g *is symbolic. Then* g *is* $\mathscr{R}$*-equivalent to the* E_k*-singularity, where*

$$\begin{aligned} E_6^{\pm} &: \ x^3\pm y^4, \\ E_7 &: \ x^3+xy^3, \\ E_8 &: \ x^3+y^5. \end{aligned}$$

Moreover, g *is simple.*

Proof. We may assume $j^3g=x^3$. We consider $\sigma=x^3$ as a 3-jet in $J^3(2,1)$. The $\mathscr{R}_1^{(4)}$-orbit of σ contains the monomials x^4, x^3y and x^2y^2. Hence a complete transversal for σ is the subspace spanned by xy^3 and y^4. By the complete transversal theorem, j^4g is $\mathscr{R}_1^{(4)}$-equivalent to $x^3+axy^3+by^4$, for some $a,b\in\mathbb{K}$. In order to simplify the notation we use the symbol $\sim$ for the $\mathscr{R}_1^{(k)}$-equivalence in $J^k(2,1)$. We have three cases:

1. $b\neq 0$. We can assume $b=\pm1$, so $j^4g\sim x^3\pm y^4+axy^3$. Use the change $(x,y)\mapsto(x,y-\frac{1}{4}ax)$ to obtain $j^4g\sim x^3\pm y^4+x^2p(x,y)$, where p is a quadratic form. Now use the change $(x,y)\mapsto(x-\frac{1}{3}p(x,y),y)$ which gives $j^4g\sim\sigma':=x^3\pm y^4$. We get $\mathfrak{m}_2^4\subset\mathfrak{m}_2J(\sigma')$ and σ' is 4-determined by the Finite Determinacy Theorem. Hence g is $\mathscr{R}$-equivalent to $E_6^{\pm}$. A simple argument shows that $E_6^{\pm}$ only deforms to $E_6^{\pm}$, A_ℓ or D_ℓ with $\ell\leq 6$. Hence $E_6^{\pm}$ is simple.

2. $b=0$ and $a\neq 0$. In this case, we can assume $a=1$, so $j^4g\sim x^3+xy^3$. We saw in Example 3.4.5 that E_7 is also 4-determined, so g is $\mathscr{R}$-equivalent to E_7. The germ E_7 is simple because it only deforms to E_7, $E_6^{\pm}$, A_ℓ or D_ℓ with $\ell\leq 7$.

3. $b=a=0$. Here $j^4g\sim\sigma':=x^3$. Since this is not 4-determined, we iterate the complete transversal method one step more. By the complete transversal theorem, $j^5g\sim x^3+dxy^4+ey^5$. If $e\neq 0$, we can assume $e=1$. After the change $(x,y)\mapsto(x,y-dx/5)$ and working modulo $\mathfrak{m}_2^2J(\sigma')=(x^4,x^3y,x^2y^2)$ we get

   $$j^5g\sim x^3+y^5$$

   which is 5-determined. Hence g is $\mathscr{R}$-equivalent to E_8. As before, it is easy to check that E_8 is simple.

   If $e=0$, then $j^5g\sim x^3+xy^4$. We proceed with the complete transversal method one step more and get that $j^6g\sim\sigma''_\lambda:=x^3+xy^4+\lambda y^6$, for some $\lambda\in\mathbb{K}$. This implies that $J(g)\subset J(\sigma''_\lambda)+\mathfrak{m}_2^6$. But then

   $$\mu(g)=\dim_{\mathbb{K}}\frac{\mathscr{O}_2}{J(g)}\geq\dim_{\mathbb{K}}\frac{\mathscr{O}_2}{J(\sigma''_\lambda)+\mathfrak{m}_2^6}=9,$$

   for all $\lambda\in\mathbb{K}$ (this last equality is left as an exercise). This contradicts the hypothesis that $\mu(g)\leq 8$ and we are done.

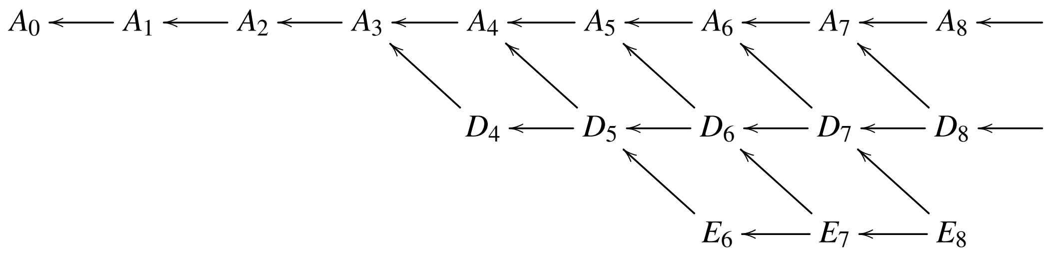
□

Corollary 6.1.8. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth such that $\mu(f) = k$ and given as in* (6.1). *We have:*

1. *If f has corank* 1*, f is $\mathscr{R}$-equivalent to A_k, $k \geq 2$.*
2. *If f has corank* 2 *and $j^3 g$ is non-degenerate or parabolic, f is $\mathscr{R}$-equivalent to D_k, $k \geq 4$.*
3. *If f has corank* 2*, $j^3 g$ is symbolic and $\mu(f) \leq 8$, f is $\mathscr{R}$-equivalent to E_k, $k = 6, 7, 8$.*

Moreover, f is simple in the three cases above.

The following diagram shows the adjacency between the ADE-singularities. An arrow from R to S means that the singularity S appears in the versal deformation of the singularity R:



Exercises

1. Suppose $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ has a Morse singularity (A_1). Show that f is simple.
2. Let $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_2^3$. Show that $\mu(g) \geq 4$.
3. Show that D_k^+ and D_k^- are not $\mathscr{R}$-equivalent when $\mathbb{K} = \mathbb{R}$ and k is odd.
4. Show that E_6^+ and E_6^- are not $\mathscr{R}$-equivalent when $\mathbb{K} = \mathbb{R}$.
5. Let $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be smooth such that $g \in \mathfrak{m}_2^3$ and $j^3 g$ is symbolic. Show that $\mu(g) \geq 6$.
6. Suppose that $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ is smooth and has corank ≥ 3. Show that $\mu(f) \geq 8$.
7. Show that any smooth germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ such that $2 \leq \mu(f) \leq 5$ is $\mathscr{R}$-equivalent to one of the seven Thom's elementary catastrophes $A_2, A_3, A_4, A_5, D_4^{\pm}, D_5$.
8. Let $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be the function $g(x, y) = x^3 + xy^4 + \lambda y^6$, for some $\lambda \in \mathbb{K}$. Show that
$$\dim_{\mathbb{K}} \frac{\mathscr{O}_2}{J(g) + \mathfrak{m}_2^6} = 9.$$
9. Suppose that $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ is smooth such that $g \in \mathfrak{m}_2^4$. Show that $\mu(g) \geq 9$.
10. Obtain the $\mathscr{R}$-classification of all smooth germs $g\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ such that $\mu(g) \leq 8$.

6.2 Non-simple singularities

The cases which are not covered by Corollary 6.1.8 are:

1. $\mu(f) = \infty$,
2. f has corank ≥ 3,
3. f has corank 2 and $j^3 g = 0$,
4. f has corank 2, $j^3 g$ is symbolic and $\mu(f) > 8$.

In the this section we will show that in such cases f is not simple, which completes the proof of the ADE-classification.

Proposition 6.2.1. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth such that $\mu(f) = \infty$. Then f is not simple.*

Proof. Since $\mathfrak{m}_n J(f) \subset J(f)$ we have $\dim_{\mathbb{K}} \mathscr{O}_n / m_n J(f) = \infty$. For each $k \geq 1$, $\mathfrak{m}_n^k \not\subset \mathfrak{m}_n J(f)$ and hence $\mathfrak{m}_n^k \not\subset \mathfrak{m}_n J(f) + \mathfrak{m}_n^{k+1}$, by Nakayama's Lemma. We choose $g_k \in \mathfrak{m}_n^k \setminus (\mathfrak{m}_n J(f) + \mathfrak{m}_n^{k+1})$.

By construction, $g_k \notin \mathfrak{m}_n J(f) + \mathfrak{m}_n^{k+1}$ and thus, $j^k g_k$ is not in the tangent space to the $\mathscr{R}^{(k)}$-orbit of $j^k f$. This implies that the 1-parameter deformation $j^k(f + t g_k)$ is not contained in the $\mathscr{R}^{(k)}$-orbit of $j^k f$. There exists $t_k \in \mathbb{K}$, with $|t_k| < 1/k$ such that $j^k(f + t_k g_k)$ is not $\mathscr{R}^{(k)}$-equivalent to $j^k f$. In particular, $f_k := f + t_k g_k$ is not $\mathscr{R}$-equivalent to f.

Moreover, f_k and f_ℓ are not $\mathscr{R}$-equivalent if $k < \ell$. Otherwise, $j^k f_k$ and $j^k f_\ell = j^k f$ would be also $\mathscr{R}^{(k)}$-equivalent, which is a contradiction. We deduce that f is not simple, since there exists an infinity family of $\mathscr{R}$-classes appearing in deformations of f. □

Proposition 6.2.2. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth of corank ≥ 3. Then f is not simple.*

Proof. By the Splitting Lemma, we have to show that any $g \in \mathfrak{m}_c^3$ with $c \geq 3$ is not simple. We work in $J^3(c, 1) \equiv \mathfrak{m}_c / \mathfrak{m}_c^4$ and consider the linear subspace $H := \mathfrak{m}_c^3 / \mathfrak{m}_c^4$ which has dimension $\binom{c+2}{3}$. For any $\sigma \in H$, $\mathscr{R}^{(3)} \sigma = \mathfrak{m}_c J(\sigma) / \mathfrak{m}^4 \subset H$. In fact, the generators of the ideal $\mathfrak{m}_c J(\sigma)$ all belong to $\mathfrak{m}_c^3$ and any element of $\mathscr{R}^{(3)} \sigma$ will be a linear combination of the classes of these generators. This implies that $\mathscr{R}^{(3)} \sigma$ has dimension at most c^2. A simple computation shows that $c^2 < \binom{c+2}{3}$ when $c \geq 3$.

Suppose g is simple. There is a finite number of $\mathscr{R}$-classes appearing in any deformation of g. Hence, there is a finite number of $\mathscr{R}^{(3)}$-classes appearing in any deformation of $\sigma_0 := j^3 g$. However, the map $H \to H$ given by $\sigma \mapsto \sigma_0 + \sigma$ is a deformation of σ_0, whose image has dimension $\binom{c+2}{3}$ and cannot covered by a finite number of $\mathscr{R}^{(3)}$-classes of dimension c^2. □

Proposition 6.2.3. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth of corank 2 such that $j^3 g = 0$. Then f is not simple.*

Proof. The proof is analogous to the proof of Proposition 6.2.2. In this case we work in $J^4(2, 1) \equiv \mathfrak{m}_2 / \mathfrak{m}_2^5$ and consider the linear subspace $H := \mathfrak{m}_2^4 / \mathfrak{m}_2^5$ which has dimension 5. We have $\sigma_0 := j^4 g \in H$ and for any $\sigma \in H$, $\mathscr{R}^{(3)} \sigma$ has dimension at most 4. Following the same argument as in the proof of Proposition 6.2.2 we deduce that g is not simple. □

Proposition 6.2.4. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth of corank 2 such that $j^3 g$ is symbolic and $\mu(f) > 8$. Then f is not simple.*

Proof. This case is a little bit more complicated. By looking at the case 3 in the proof of Proposition 6.1.7, we can assume that $j^5 g = x^3 + xy^4$. This implies that $g \in (x^3, xy^4) + \mathfrak{m}_2^6 \subset I := (x, y^2)^3$. We work in $J^6(2,1) \equiv \mathfrak{m}_2/\mathfrak{m}_2^7$ and consider the linear subspace $H := I/\mathfrak{m}_2^7$ which has dimension 16. We remark that H is not $\mathscr{R}^{(6)}$-invariant.

Given $h \in \mathscr{O}_2 \setminus \{0\}$, we denote by $\mathrm{ord}(h)$ the weighted order of h with respect to the weights $(2,1)$ of the variables (x,y). If $h \in I$ then $\mathrm{ord}(h) \geq 6$ and hence the four generators of $\mathfrak{m}_2 J(h)$, if not zero, have weighted order:

$$\mathrm{ord}\left(x\frac{\partial h}{\partial x}\right) \geq 6,\ \mathrm{ord}\left(y\frac{\partial h}{\partial x}\right) \geq 5,\ \mathrm{ord}\left(x\frac{\partial h}{\partial y}\right) \geq 7,\ \mathrm{ord}\left(y\frac{\partial h}{\partial y}\right) \geq 6.$$

In particular, there are at most three linearly independent elements in $\mathfrak{m}_2 J(h)$ of weighted order six, namely, $x\frac{\partial h}{\partial x}$, $y^2\frac{\partial h}{\partial x}$ and $y\frac{\partial h}{\partial y}$. But I has four linearly independent elements of weighted order six, so $I \not\subset \mathfrak{m}_2 J(h)$.

The above argument shows that for any $\sigma \in H$, H is not contained in $T_\sigma \mathscr{R}^{(6)}\sigma$. The set $H \cap \mathscr{R}^{(6)}\sigma$ is constructible in H when $\mathbb{K} = \mathbb{C}$ or semialgebraic when $\mathbb{K} = \mathbb{R}$. In both cases, if $H \cap \mathscr{R}^{(6)}\sigma$ had dimension 16, then it would contain non-empty open subset of H and hence H would be contained in $T_\sigma \mathscr{R}^{(6)}\sigma$. Thus, $H \cap \mathscr{R}^{(6)}\sigma$ must have dimension at most 15.

Finally, we proceed as in the proofs of Propositions 6.2.2 and 6.2.3. In fact, H cannot be covered by a finite number of $\mathscr{R}^{(6)}$-classes of elements in H and this implies that g is not simple. □

This completes the proof of the ADE-classification. In the next corollary we also include the A_k-notation in the cases that f is regular (A_0) or Morse (A_1).

Corollary 6.2.5 (Arnold's ADE-classification)**.** *A smooth function germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ is simple if and only if it is $\mathscr{R}$-equivalent to A_k, D_k, E_6, E_7 or E_8.*

Additional remarks on modality

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth with $\mu := \mu(f)$, such that $0 < \mu < \infty$. The $\mathscr{R}$-codimension of f is $r = \mu + n - 1$ by Proposition 2.1.10 and hence, we have an r-parameter $\mathscr{R}$-miniversal deformation $F(x,u) = f_u(x)$.

The *bifurcation set* $\mathcal{B}$ is defined as the set germ of parameters u in $\mathbb{K}^r$ close to 0 such that the germ f_u is not regular. In fact, $\mathcal{B}$ is a smooth submanifold of dimension $\mu - 1$, defined by the vanishing of the partial derivatives of f_u at the origin.

The *moduli space* $\mathcal{M}$ is defined as the quotient space of $\mathcal{B}$ under the relation $u \sim u'$ if f_u is $\mathscr{R}$-equivalent to $f_{u'}$. The *modality* is the dimension of the moduli space $\mathcal{M}$. Roughly speaking, it is the minimal number of parameters we need to parametrise all the $\mathscr{R}$-classes appearing in the versal deformation.

It follows that f is simple if and only if it has modality 0. For instance, it can be shown that the singularity $f_t(x,y) = (x+ty)y(x-y)(x+y)$ of Example 6.1.3 is unimodal (i.e., it has modality 1). In fact, the normal form for this singularity in Arnold's notation is X_9, given by $x^4 + y^4 + ax^2y^2$, with $a^2 \neq 4$ (see Example 6.2.8 below).

A celebrated theorem due to Gabrielov [9, Theorem 6] alllows to compute the modality in an easier way in the complex case $\mathbb{K} = \mathbb{C}$. The μ = *constant stratum* is defined as the subset of parameters u in the bifurcation set $\mathcal{B}$ such that $\mu(f_u) = \mu(f)$.

Theorem 6.2.6 (Gabrielov)**.** *The modality is equal to the dimension of the μ = constant stratum.*

When f is weighted homogeneous (see Definition 7.2.9 in Section 7.2), the dimension of the μ = constant stratum is easy to compute thanks to the following result due to Varchenko [30, Theorem 2]:

Theorem 6.2.7 (Varchenko)**.** *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be weighted homogeneous of type $(\boldsymbol{w}; d)$ with isolated critical point. Let $\{g_1, \dots, g_\mu\}$ be a monomial basis of $\mathscr{O}_n/J(f)$ with $g_1 = 1$ and consider the associated versal unfolding*

$$F(x, u) = f_u(x) = f(x) + \sum_{i=1}^{\mu} u_i g_i(x).$$

Then the μ = constant stratum is given by the equations $u_i = 0$ such that g_i has weighted degree $< d$.

It follows that the dimension of the μ = constant stratum (and hence the modality) is the number of monomials g_i of weighted degree $\geq d$.

Example 6.2.8. Consider the X_9 singularity $x^4 + y^4 + ax^2y^2$, with $a^2 \neq 4$. This is weighted homogeneous of type $(1, 1; 4)$ and a monomial basis of $\mathscr{O}_2/J(f)$ is

$$1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2.$$

We find one monomial x^2y^2 of degree 4, so X_9 is unimodal by Theorems 6.2.6 and 6.2.7.

Exercises

1. Use Theorems 6.2.6 and 6.2.7 to give another proof that the ADE-singularities are simple.

2. Consider the J_{10} singularity $x^3+y^6+ax^2y^2$, with $4a^3+27 \neq 0$ (this is the same singularity as $x^3 + xy^4 + \lambda y^6$ in the proof of Proposition 6.1.7, case 3). Show that J_{10} is unimodal.

Chapter 7

The Milnor number

In this chapter we will study some important properties of the Milnor number $\mu(f)$. In Section 7.1 we will see the conservation of the Milnor number and some interesting consequences. In Section 7.2 we will introduce another important invariant, namely, the Tjurina number $\tau(f)$ and how it is related to $\mu(f)$. Moreover we also define weighted homogeneous functions, which play an important role in this context. Finally, in Section 7.3 we will explain the relationship between $\mu(f)$ and the Milnor fibration in the complex case, which makes clear the terminology of Milnor number for $\mu(f)$.

7.1 Conservation of the Milnor number

On of the main properties of the Milnor number is its conservation under deformation in the complex case property. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic with isolated critical point and let $F(x, u) = f_u(x)$ be any deformation (non necessarily origin preserving). We fix a representative $F\colon X \times U \to \mathbb{C}$, where X and U are open neighbourhoods of the origin in $\mathbb{C}^n$ and $\mathbb{C}^r$, respectively. For each $u \in U$ we consider the function $f_u\colon X \to \mathbb{C}$.

Proposition 7.1.1 (Conservation of the Milnor number)**.** *With the above notation, for X and U small enough and for all $u \in U$ we have*

$$\mu(f) = \sum_{x \in \Sigma(f_u)} \mu_x(f_u),$$

where $\Sigma(f_u)$ is the set of critical points of f_u and $\mu_x(f_u)$ is the Milnor number of the germ of f_u at x.

Proof. The relative Jacobian ideal $J_{rel}(F)$ is the ideal in $\mathscr{O}_{n+r}$ generated by the partial derivatives of F with respect to $x_1, \dots, x_n$. On one hand, $\dim V(J_{rel}(F)) \geq r$ since $J_{rel}(F)$ is generated by n elements. On the other hand, the projection $\pi\colon V(J_{rel}(F)) \to (\mathbb{C}, 0)$ is finite, because $\pi^{-1}(0) = V(J(f)) \subset \{0\}$. By Remmert's Finite Mapping theorem (see [14, §5, Chapter V]) the image $\pi(V(J_{rel}(F))$ is analytic in $(\mathbb{C}^r, 0)$ and has the same dimension as $V(J_{rel}(F))$, so $\dim V(J_{rel}(F)) \leq r$. Thus, $\dim V(J_{rel}(F)) = r$.

This implies that $\mathscr{O}_{n+r}/J_{rel}(F)$ is a complete intersection and hence, Cohen-Macaulay. By a well known theorem in complex analytic geometry we have conservation of multiplicity (see

[6, Theorem 6.4.7]), which means that for X and U small enough and for all $u \in U$ we have

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_n}{J(f)} = \sum_{x \in V(J(f_u))} \dim_{\mathbb{C}} \frac{\mathscr{O}_{n,x}}{J(f_u)_x},$$

where $\mathscr{O}_{n,x}$ is the ring of function germs $(\mathbb{C}^n, x) \to \mathbb{C}$ and $J(f_u)_x$ is the Jacobian ideal of f_u at x. This gives

$$\mu(f) = \sum_{x \in \Sigma(f_u)} \mu_x(f_u).$$

□

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth. By Proposition 1.3.6 we have that $\mu(f) < \infty$ if and only if $\mathfrak{m}_n^k \subseteq J(f)$, for some k. A consequence of this is that when f is real analytic then the Milnor numbers of f and its complexification $f_{\mathbb{C}}$ coincide.

Proposition 7.1.2. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic. Then $\mu(f) < \infty$ if and only if $\mu(f_{\mathbb{C}}) < \infty$, where $f_{\mathbb{C}}$ is the complexification of f. Moreover, if $\mu(f) < \infty$ then $\mu(f) = \mu(f_{\mathbb{C}})$.*

Proof. In order to distinguish between the real and complex cases, we use here the notations $(\mathscr{E}_n, \mathfrak{m}_n)$ and $(\mathscr{O}_n, \mathfrak{m}'_n)$ for the local rings of C^∞-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ and holomorphic germs $(\mathbb{C}^n, 0) \to \mathbb{C}$, respectively.

If $\mu(f) < \infty$, then $\mathfrak{m}_n^k \subset J(f)$ for some k. This implies that for any multi-index α, with $|\alpha| = k$,

$$x^\alpha = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i},$$

for some $a_i \in \mathscr{E}_n$. Replacing each a_i by its k-jet, we get a polynomial version of these equalities modulo $\mathfrak{m}_n^{k+1}$. These equalities are also valid if we take the corresponding complexifications, giving the inclusion ${\mathfrak{m}'_n}^k \subset J(f_{\mathbb{C}}) + {\mathfrak{m}'_n}^{k+1}$. By Nakayama's Lemma, ${\mathfrak{m}'_n}^k \subset J(f_{\mathbb{C}})$ and hence, $\mu(f_{\mathbb{C}}) < \infty$.

Conversely, if $\mu(f_{\mathbb{C}}) < \infty$, then ${\mathfrak{m}'_n}^k \subset J(f_{\mathbb{C}})$. As above, for each α, with $|\alpha| = k$,

$$x^\alpha = \sum_{i=1}^n a_i \frac{\partial f_{\mathbb{C}}}{\partial x_i},$$

for some $a_{ij} \in \mathscr{O}_n$. Since the partial derivatives have real coefficients, the same equalities hold with a_i^*, the function obtained from a_i by complex conjugating the coefficients in its Taylor expansion, in place of a_i. It follows that

$$x^\alpha = \frac{1}{2} \sum_{i=1}^n (a_i + a_i^*) \frac{\partial f_{\mathbb{C}}}{\partial x_i},$$

where now the functions $a_i + a_i^*$ have real coefficients. We deduce that $\mathfrak{m}_n^k \subset J(f)$ and thus, $\mu(f) < \infty$.

For the second part, take k such that $\mathfrak{m}_n^k \subset J(f)$ and ${\mathfrak{m}'_n}^k \subset J(f_{\mathbb{C}})$. The polynomial complexification gives an isomorphism

$$\frac{\mathscr{E}_n}{\mathfrak{m}_n^k} \otimes_{\mathbb{R}} \mathbb{C} \cong \frac{\mathscr{O}_n}{{\mathfrak{m}'_n}^k},$$

which maps $(J(f)/\mathfrak{m}_n^k) \otimes_{\mathbb{R}} \mathbb{C}$ onto $J(f_{\mathbb{C}})/{\mathfrak{m}'_n}^k$. Therefore, $\mu(f) = \mu(f_{\mathbb{C}})$. □

The first application of the conservation of the Milnor number is that $\mu(f)$ is upper semi-continuous, which also holds in the real C^∞-case.

Corollary 7.1.3 (Upper semicontinuity)**.** *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth such that $\mu(f) < \infty$ and let $F(x, u) = f_u(x)$ any deformation. For all (x, u) in a neighbourhood of the origin, we have $\mu(f) \geq \mu_x(f_u)$.*

Proof. The result is obvious in the complex case from the conservation of Milnor number, Proposition 7.1.1. When $\mathbb{K} = \mathbb{R}$, we first prove the result in the case that F is the versal deformation of f. By the Finite Determinacy Theorem, we can assume that f and F are both polynomial. We denote by $f_{\mathbb{C}}$ and $(f_u)_{\mathbb{C}}$ the complexifications of f and f_u respectively. By Proposition 7.1.2, $\mu(f) = \mu(f_{\mathbb{C}})$ and $\mu_x(f_u) = \mu_x((f_u)_{\mathbb{C}})$, so the real case follows from the complex one.

Now, let $G(x, v) = g_v(x)$ be any deformation of f. By versality, there exist a diffeomorphism $\Phi\colon (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^n \times \mathbb{R}^s, 0)$ which is an unfolding of the identity and a base change map $h\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$ such that $G \circ \Phi = h^* F$. In particular, $g_v \circ \phi_v = f_{h(v)}$, so the germ of g_v at x is $\mathscr{R}$-equivalent to the germ of $f_{h(v)}$ at $\phi_v^{-1}(x)$ and we are done.

□

The second application of the conservation of the Milnor number is that, in the complex case, $\mu(f)$ is equal to the number of critical points which appear in a Morsification of f.

Definition 7.1.4. Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth and let $F(x, t) = f_t(x)$ be a 1-parameter deformation. We say F is a *Morsification* if there exists a representative $F\colon X \times T \to \mathbb{K}$, where X and T are open neighbourhoods of the origin in $\mathbb{K}^n$ and $\mathbb{K}$, respectively, such that $f_t\colon X \to \mathbb{K}$ is a Morse function, for all $t \in \mathbb{K} \setminus \{0\}$.

Proposition 7.1.5. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth such that $\mu(f) < \infty$. Then a generic choice of $a \in \mathbb{K}^n$ gives a Morsification F defined by*

$$F(x, t) = f_t(x) = f(x) + t(a_1 x_1 + \ldots a_n x_n).$$

Proof. We consider first the n-parameter deformation

$$H(x, a) = h_a(x) = f(x) + a_1 x_1 + \ldots a_n x_n.$$

For each $a \in \mathbb{K}^n$, the gradient and the Hessian of h_a are, respectively, $\nabla h_a(x) = \nabla f(x) + a$ and $Hh_a(x) = Hf(x)$. Thus, h_a is a Morse function if and only if $-a$ is a regular value of $\nabla f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$. We know from Sard's theorem that the set C of critical values of ∇f is a null set in $\mathbb{K}^n$.

When $\mathbb{K} = \mathbb{C}$ the set of critical points of ∇f is analytic. Moreover, f has isolated critical point and hence $(\nabla f)^{-1}(0) = \{0\}$. By Remmert's Finite Mapping theorem C is also analytic in $(\mathbb{C}^n, 0)$, necessarily of dimension at most $n - 1$. Otherwise, if $\mathbb{K} = \mathbb{R}$ we can assume, by the Finite Determinacy theorem, that f is a polynomial. The set of critical points of ∇f is now algebraic and hence, its image C is semialgebraic in $(\mathbb{K}^n, 0)$, by the Tarski-Seidenberg theorem (see for instance [13]). Since C is a null set, $\dim C \leq n - 1$ and also its Zariski closure $Z(C)$ has dimension at most $n - 1$.

In both cases, C is contained in a hypersurface $V(H)$, for some $H \in \mathscr{O}_n$, $H \neq 0$ and H is polynomial if $\mathbb{K} = \mathbb{R}$. After a generic linear change of variables we can can assume that H

is regular in one of the variables, say a_n (see [6, Lemma 2.1.6] in the case H is polynomial or [6, Lemma 3.2.2] in the case H is holomorphic). This means that $H(0, \dots, 0, a_n) \neq 0$ and hence, $C \cap L = \{0\}$, where L is the a_n-axis. This shows that $C \cap L = \{0\}$ for any generic line $L \in \mathbb{K}^n$ through the origin. If L is parametrised by $t \mapsto ta$, then $f_t = h_{ta}$ gives the desired Morsification. □

The following corollary is a direct consequence of the conservation of Milnor number Proposition 7.1.1, taking into account that $\mu_x(f_t) = 1$ when f_t has a Morse critical point at x.

Corollary 7.1.6. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic with isolated critical point and let $F(x,t) = f_t(x)$ be any Morsification of f. There exists a representative $F\colon X \times T \to \mathbb{C}$ such that $\mu(f)$ is equal to the number of critical points of $f_t\colon X \to \mathbb{C}$, for all $t \in T \setminus \{0\}$.*

Let $\phi\colon (M, x) \to (N, x)$ be a C^∞ map germ between oriented C^∞-manifolds M and N such that $\phi^{-1}(y) = \{x\}$ (as set germs). Then ϕ has a well defined *local degree* denoted by $\deg(\phi)$. This is defined by considering a representative $\phi\colon U \to V$, where U and V are open neighbourhoods of x and y in M and N, respectively, such that $\phi^{-1}(y) = \{x\}$ (as sets), ϕ is proper and V is connected. Then $\deg(\phi)$ is defined as the (global) degree of $\phi\colon U \to V$ (i.e., the number of preimages of a regular value taking into account the sign of the Jacobian determinant).

Corollary 7.1.7. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic with isolated critical point. Then $\mu(f) = \deg(\nabla f)$, where $\nabla f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is the gradient of f.*

Proof. Let $F(x,t) = f_t(x)$ be any Morsification of f. We take representatives $F\colon X \times T \to \mathbb{C}$ as in Corollary 7.1.6 and $\nabla f\colon X \to V$ and as in the previous paragraph, so that its (global) degree is the local degree $\deg(\nabla f)$. It is well known that the (global) degree is invariant under homotopy. Hence, $\deg(\nabla f) = \deg(\nabla f_t)$, for the gradient map $\nabla f_t\colon X \to V$ and for all $t \in T$.

When $t \neq 0$, f_t is a Morse function, so 0 is a regular value of ∇f_t. Moreover, x is a preimage of 0 by ∇f_t if and only if x is a critical point of f_t. Since ∇f_t is holomorphic, its real Jacobian determinant is always > 0 at any critical point x. This shows that $\deg(\nabla f_t)$ is equal to the number of critical points of f_t which is equal to $\mu(f)$, by Corollary 7.1.6. □

In the real case, we also have the following corollary whose proof is analogous to that of Corollaries 7.1.6 and 7.1.7. Details are left as an exercise.

Corollary 7.1.8. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be smooth such that $\mu(f) < \infty$. Then:*

1. $|\deg(\nabla f)| \leq \mu(f)$ *and* $\deg(\nabla f) \equiv \mu(f) \mod 2$, *where $\nabla f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the gradient of f.*
2. *If $F(x,t) = f_t(x)$ is a Morsification of f then the number of critical points of f_t for $t \neq 0$ is $\leq \mu(f)$ and also congruent modulo 2 to $\mu(f)$.*

Exercises

1. Let $I \subset \mathcal{O}_n$ be such that $\dim_{\mathbb{K}} \mathcal{O}_n/I < \infty$. Show that $V(I) \subset \{0\}$, where $V(I)$ is the set germ in $(\mathbb{K}^n, 0)$ given by the zeros of I (i.e., the points x in a neighbourhood of 0 in $\mathbb{K}^n$ such that $h(x) = 0$ for all $h \in I$).

2. Assume $\mathbb{K} = \mathbb{C}$ and let $I \subset \mathscr{O}_n$ be such that $V(I) \subset \{0\}$. Show that $\dim_{\mathbb{K}} \mathscr{O}_n/I < \infty$. Hint: Use the Rückert Nullstellensatz (the complex analytic version of the Hilbert Nullstellensatz, see for instance [6, Theorem 3.4.4]).

3. Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be the $D_k^{\pm}$-singularity. Find the set of pairs $(a, b) \in \mathbb{K}^2$ such that $f_{a,b}\colon \mathbb{K}^2 \to \mathbb{K}$ is a Morse function, where

$$f_{a,b}(x, y) = f(x, y) + ax + by.$$

4. Show Corollary 7.1.8.

7.2 Milnor vs. Tjurina

Given $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ smooth, the $\mathscr{K}_e$-codimension of f is also known as the *Tjurina number* and is denoted by $\tau(f)$, that is,

$$\tau(f) := \mathscr{K}_e - \operatorname{codim}(f) = \dim_{\mathbb{K}} \frac{\mathscr{O}_n}{J(f) + (f)}.$$

In the second part of this course, we will see that two map germs $f, g\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are $\mathscr{K}$-equivalent if and only if there exists a diffeomorphism $\phi\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\phi^*(f_1, \dots, f_p) = (g_1, \dots, g_p)$, where $\phi^*\colon \mathscr{O}_n \to \mathscr{O}_n$ is the induced ring isomorphism. The following lemma says that the $\mathscr{K}_e$-codimension is invariant under $\mathscr{K}$-equivalence. This is true in general, but for the moment we prove it only in the case $p = 1$.

Lemma 7.2.1. *Let $f, g\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be smooth and assume there exists a diffeomorphism $\phi\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\phi^*(f) = (g)$. Then $\tau(f) = \tau(g)$.*

Proof. Define $h := f \circ \phi = \phi^*(f)$. From the $\mathscr{R}$-invariance of the Milnor number we know that $\phi^*(J(f)) = J(h)$ and hence, $\phi^*(J(f) + (f)) = J(h) + (h)$. This shows that $\tau(f) = \tau(h)$. Now we have $(h) = (g)$, which is equivalent to $h = \lambda g$, for some $\lambda \in \mathscr{O}_n$ such that $\lambda(0) \neq 0$. For all $i = 1, \dots, n$,

$$\frac{\partial h}{\partial x_i} = \frac{\partial \lambda}{\partial x_i} g + \lambda \frac{\partial g}{\partial x_i} \in J(g) + (g),$$

We have $J(h) + (h) \subseteq J(g) + (g)$. Since $g = \frac{1}{\lambda} h$, we also have the opposite inclusion. Thus, $J(h) + (h) = J(g) + (g)$, which implies $\tau(h) = \tau(g)$. □

Let $\mathbb{K} = \mathbb{C}$. We recall that a *hypersurface* is an analytic set germ $(X, 0)$ in $(\mathbb{C}^n, 0)$ defined by the zeros of some non-constant holomorphic function $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. By the Rückert Nullstellensatz [6, Theorem 3.4.4], the ideal $I(X, 0)$ of functions vanishing on $(X, 0)$ is equal to $\sqrt{(f)} = (\tilde{f})$, where $\tilde{f}$ is obtained from f after eliminating repeated irreducible factors. Such a function $\tilde{f}$ is called a *reduced equation* for $(X, 0)$. The *local ring* of $(X, 0)$ is the quotient ring $\mathscr{O}_{X,0} := \mathscr{O}_n/(\tilde{f})$.

Definition 7.2.2. Let $(X, 0)$ be a hypersurface in $(\mathbb{C}^n, 0)$. The *Tjurina number* of $(X, 0)$ is defined as $\tau(X, 0) = \tau(f)$, where f is a reduced equation of $(X, 0)$.

By Lemma 7.2.1, $\tau(X, 0)$ is well defined, that is, it is independent of the choice of the reduced equation. In fact, we have more:

Lemma 7.2.3. *Let $(X,0)$ be a hypersurface in $(\mathbb{C}^n,0)$. Then:*

1. $\tau(X,0)=0$ *if and only if* $(X,0)$ *is smooth;*
2. $\tau(X,0)<\infty$ *if and only if* $(X,0)$ *has isolated singularity;*
3. $\tau(X,0)$ *only depends on the isomorphism class of its local ring* $\mathcal{O}_{X,0}$.

Proof. 1. and 2. follow from the fact that $V(J(f)+(f))$ is the set germ of singular points of $(X,0)$ with reduced equation f (see Exercises 7.1.1 and 7.1.2).

3. Let $(X,0)$ and $(Y,0)$ be hypersurfaces in $(\mathbb{C}^n,0)$ with reduced equations f and g, respectively. Suppose there exists an isomorphism $A\colon \mathcal{O}_{X,0}\to\mathcal{O}_{Y,0}$. If f is regular, then $\mathcal{O}_{X,0}\cong\mathcal{O}_{n-1}$ and hence, $\mathcal{O}_{Y,0}\cong\mathcal{O}_{n-1}$, so g is also regular. We have $\tau(f)=\tau(g)=0$.

Thus, we can assume that f and g are both singular, that is, $f,g\in\mathfrak{m}_n^2$. For each $i=1,\dots,n$ we have $A([x_i])=([\phi_i])$, for some $\phi_i\in\mathfrak{m}_n$. Let $\mathfrak{m}_{X,0}=\mathfrak{m}_n/(f)$ and $\mathfrak{m}_{Y,0}=\mathfrak{m}_n/(g)$ be the maximal ideals of $\mathcal{O}_{X,0}$ and $\mathcal{O}_{Y,0}$, respectively. The isomorphism A induces a $\mathbb{C}$-linear isomorphism

$$\frac{\mathfrak{m}_{X,0}}{\mathfrak{m}_{X,0}^2}\equiv\frac{\mathfrak{m}_n}{\mathfrak{m}_n^2}\longrightarrow\frac{\mathfrak{m}_{Y,0}}{\mathfrak{m}_{Y,0}^2}\equiv\frac{\mathfrak{m}_n}{\mathfrak{m}_n^2},$$

which sends the class of x_i into the class of ϕ_i. In fact, such isomorphism is nothing but the differential at the origin of the map germ $\phi:=(\phi_1,\dots,\phi_n)\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$. By the inverse mapping theorem, ϕ is a diffeomorphism.

Consider the diagram

$$\begin{array}{ccc}\mathcal{O}_n & \xrightarrow{\phi^*} & \mathcal{O}_n\\ \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi_2}\\ \mathcal{O}_{X,0} & \xrightarrow{A} & \mathcal{O}_{Y,0}\end{array}$$

where the vertical arrows are the quotient mappings. We have $\pi_2(\phi^*(x_i))=A(\pi_1(x_i))$ for all $i=1,\dots,n$. Hence $\pi_2(\phi^*(h))=A(\pi_1(h))$, for any function $h\in\mathcal{O}_n$ which is a polynomial. Given any function $h\in\mathcal{O}_n$, for all $k\geq 1$ we have $h=j^kh$ modulo $\mathfrak{m}_n^{k+1}$. This implies

$$\pi_2(\phi^*(h))-A(\pi_1(h))\in\mathfrak{m}_{Y,0}^{k+1}.$$

and hence,

$$\pi_2(\phi^*(h))-A(\pi_1(h))\in\bigcap_{k=1}^{\infty}\mathfrak{m}_{Y,0}^{k+1}=\{0\}.$$

This shows that the diagram is commutative. In particular, $\phi^*(f)=(g)$, so $\tau(f)=\tau(g)$ by Lemma 7.2.1. □

The following theorem is known as the Łojasiewicz inequality for analytic functions (either real or complex). It is a useful tool to give a closer relationship between the Tjurina and the Milnor numbers.

Theorem 7.2.4 (Łojasiewicz inequality)**.** *Let $f\colon(\mathbb{K}^n,0)\to(\mathbb{K},0)$ be analytic. There exist a neighbourhood U of 0 in $\mathbb{K}^n$ and real numbers $C>0\cdot$ and $0<\theta<1$ such that*

$$|\nabla f(x)|\geq C|f(x)|^{\theta},\tag{7.1}$$

for all $x\in U$.

Proof. See [14, Proposition 1, p. 67] □

Corollary 7.2.5. *Let* $f\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ *be holomorphic. Then* $\mu(f)<\infty$ *if and only if* $\tau(f)<\infty$.

Proof. It follows from the definitions that $\tau(f)\le\mu(f)$. In particular, $\tau(f)<\infty$ when $\mu(f)<\infty$. For the converse we use the Łojasiewicz inequality, Theorem 7.2.4. In fact, (7.1) implies

$$V(J(f))\subseteq V(J(f)+(f)).$$

If $\tau(f)<\infty$, then $V(J(f)+(f))\subseteq\{0\}$ by Lemma 7.2.3. Hence $V(J(f))\subseteq\{0\}$ and thus, $\mu(f)<\infty$ by Proposition 3.2.1. □

Another application of Łojasiewicz inequality is that the Milnor number is in fact an invariant of the hypersurface $(X,0)$, provided it has isolated singularity.

Proposition 7.2.6. *Let* $(X,0)$ *be a hypersurface with isolated singularity in* $(\mathbb{C}^n,0)$. *If* f *and* g *are reduced equations of* $(X,0)$, *then* $\mu(f)=\mu(g)$.

Proof. The result is obvious when $(X,0)$ is smooth, so we suppose 0 is a critical point of f. We have $g=\lambda f$, for some $\lambda\in\mathscr{O}_n$ such that $\lambda(0)\neq 0$. Without loss of generality we can assume that $\lambda(0)=1$. We consider the family

$$F(x,t)=f_t(x)=\lambda_t(x)f(x),$$

where $\lambda_t(x)=1+t(\lambda(x)-1)$ and $t\in\mathbb{C}$.

By the the Łojasiewicz inequality, Theorem 7.2.4, there exist a neighbourhood U of 0 in $\mathbb{C}^n$ and real numbers $C>0\cdot$ and $0<\theta<1$, for which (7.1) holds. Since $\mu(f)<\infty$, we can also suppose that 0 is the only critical point of f on U. For all $(x,t)\in U\times\mathbb{C}$,

$$\begin{aligned}|\nabla f_t(x)|&=|\nabla\lambda_t(x)f(x)+\lambda_t(x)\nabla f(x)|\\&\ge|\lambda_t(x)||\nabla f(x)|-|\nabla\lambda_t(x)||f(x)|\\&\ge|\lambda_t(x)||\nabla f(x)|-C^{-\frac{1}{\theta}}|\nabla f(x)|^{\frac{1}{\theta}}\\&=|\nabla f(x)|\rho(x,t),\end{aligned}$$

where

$$\rho(x,t):=|\lambda_t(x)|-|\nabla\lambda_t(x)|C^{-\frac{1}{\theta}}|\nabla f(x)|^{\frac{1}{\theta}-1}.$$

The function $\rho\colon U\times\mathbb{C}\to\mathbb{R}$ is continuous and for each $t_0\in\mathbb{C}$, $\rho(0,t_0)=1$. There exist $U'\subseteq U$ open neighbourhood of 0 and T open neighbourhood of t_0 in $\mathbb{C}$ such that $|\rho(x,t)|>0$, for all $(x,t)\in U'\times T$. Hence, 0 is the only critical point of f_t on U', for all $t\in T$. By the conservation of the Milnor number, Proposition 7.1.1, $\mu(f_t)=\mu(f_{t_0})$, for all $t\in T$. This shows that $\mu(f_t)$ is locally constant on $t\in\mathbb{C}$. Since $\mathbb{C}$ is connected, $\mu(f_t)$ is (globally) constant on $t\in\mathbb{C}$. In particular, $\mu(f)=\mu(f_0)=\mu(f_1)=\mu(g)$. □

Definition 7.2.7. Let $(X,0)$ be a hypersurface in $(\mathbb{C}^n,0)$ with isolated singularity. The *Milnor number* of $(X,0)$ is defined as $\mu(X,0)=\mu(f)$, where f is a reduced equation of $(X,0)$.

Remark 7.2.8. When $(X, 0)$ is a hypersurface in $(\mathbb{C}^n, 0)$ (not necessarily with isolated singularity), we can always consider the Tjurina algebra $\mathscr{O}_n/(J(f) + (f))$ and the Milnor algebra $\mathscr{O}_n/J(f)$, where f is a reduced equation of $(X, 0)$. It follows from the proof of Lemma 7.2.1 that the isomorphism class of the Tjurina algebra is independent of the choice of the reduced equation f. However, in general, we may find non isomorphic Milnor algebras for different choices of f in the non-isolated singularity case.

From the definitions of $\tau(f)$ and $\mu(f)$ it follows easily that $\tau(f) \leq \mu(f)$, with equality if and only if $f \in J(f)$ (provided that $\mu(f) < \infty$). We see in the last part of this section that such condition is related to the fact that f is weighted homogeneous.

Let $\mathbf{w} = (w_1, \dots, w_n)$ be an n-tuple of positive real numbers, which we call *weights*. The *weighted degree* with respect to $\mathbf{w}$ of a monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is defined as $w_1\alpha_1 + \cdots + w_n\alpha_n$.

Definition 7.2.9. We say that $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ is *weighted homogeneous* of type $(\mathbf{w}; d)$ if it is a polynomial and all its monomials have weighted degree d with respect to $\mathbf{w}$.

When $\mathbb{K} = \mathbb{C}$, we say that a hypersurface $(X, 0)$ in $(\mathbb{C}^n, 0)$ is *weighted homogeneous* of type $(\mathbf{w}; d)$ if it has a reduced equation f which is weighted homogeneous of type $(\mathbf{w}; d)$.

Example 7.2.10. The condition that a function is weighted homogeneous is not invariant under $\mathscr{R}$-equivalence in general. In fact, it is not invariant under linear change of coordinates. For instance, the function $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ given by $f(x, y) = x^2 + y^3$ is weighted homogeneous of type $(3, 2; 6)$. However, if we perform the linear change $(x, y) \mapsto (x + y, y)$ we get $g(x, y) = x^2 + 2xy + y^2 + y^3$ which is not weighted homogeneous of any type.

The following two properties are easily deduced from the definition. The proof is left as exercise.

Proposition 7.2.11. *Let f be weighted homogeneous of type $(\mathbf{w}; d)$. Then:*

1. *For all $x \in \mathbb{K}^n$ and $\lambda \in \mathbb{K}$,*

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \dots, x_n),$$

2. *The* Euler identity*:*

$$f = \frac{1}{d}\left(w_1 x_1 \frac{\partial f}{\partial x_1} + \cdots + w_n x_n \frac{\partial f}{\partial x_n}\right).$$

The Euler identity implies in particular that $f \in J(f)$ and hence, $\tau(f) = \mu(f)$.

Corollary 7.2.12. *If $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ is weighted homogeneous then $\tau(f) = \mu(f)$.*

When $\mathbb{K} = \mathbb{C}$ and f has isolated critical point, Saito [23] showed that the equality $\tau(f) = \mu(f)$ implies weighted homogeneity, up to $\mathscr{R}$-equivalence.

Theorem 7.2.13 (Saito)**.** *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic with isolated critical point. Then $f \in J(f)$ if and only if f is $\mathscr{R}$-equivalent to a weighted homogeneous function.*

Finally, we give a formula due to Milnor and Orlik [19] for the Milnor number (and the Tjurina number) of a weighted homogeneous singularity in terms of the weights and the degree.

Theorem 7.2.14 (Milnor-Orlik)**.** *Let* $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ *be weighted homogeneous of type* $(\mathbf{w}; d)$*. Then,*

$$\mu(f) = \frac{(d - w_1)\dots(d - w_n)}{w_1 \dots w_n}.$$

Proof. Since f is weighted homogeneous of type $(\mathbf{w}; d)$, each partial derivative $\partial f/\partial x_i$ is weighted homogeneous of type $(\mathbf{w}; d - w_i)$. The formula for $\mu(f) = \dim_{\mathbb{K}} \mathscr{O}_n/J(f)$ follows directly from the weighted version of Bezout's Theorem (see for instance [2, Lemma 5.6]). □

Exercises

1. Show that all simple germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ are weighted homogeneous and say which are the weights and the degrees in each case.

2. Show Proposition 7.2.11.

7.3 Relationship with the Milnor fibration

The Milnor fibration of a holomorphic function germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ was introduced by Milnor in his famous book [18]. This fibration will be explained with more detail in the Course B: Topology of complex singularities. Here we just present the main theorem which will make clear the terminology of Milnor number that we have used along this course.

Take a representative $f\colon U \to \mathbb{C}$, where U is an open neighbourhood of 0 in $\mathbb{C}^n$. We denote by B_ϵ the closed ball in $\mathbb{C}^n$ of radius ϵ centered at 0 and by D_δ the closed disk in $\mathbb{C}$ of radius $\delta > 0$, also centered at 0.

Theorem 7.3.1. *For any sufficiently small* $\epsilon > 0$ *and any* $0 < \delta \ll \epsilon$*, the restriction*

$$f\colon B_\epsilon \cap f^{-1}(D_\delta \setminus \{0\}) \longrightarrow D_\delta \setminus \{0\}$$

is a locally trivial C^∞ *fibration, whose fibre* F *is called the Milnor fibre. The isotopy type of the fibration is independent of the choice of* ϵ *and* δ*. In case* f *has isolated critical point, its Milnor fibre* F *has the homotopy type of a bouquet of spheres, that is,*

$$F \overset{ht}{\simeq} S^{n-1} \vee \dots \vee S^{n-1}$$

and the number of such spheres is called the Milnor number and is denoted by $\mu(f)$*.*

One of the proofs that the Milnor fibre F has the homotopy type of a bouquet of spheres is based on Morse theory. One shows that the number $\mu(f)$ of spheres is equal to the number of critical points in a Morsification f_t of f. By Corollary 7.1.6, $\mu(f)$ coincides with our definition of Milnor number, namely,

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathscr{O}_n}{J(f)}.$$

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