Lecture 4: Equisingularity and ICIS

by Terence Gaffney

- Welcome to Equisingularity!
- Please read the notes for lecture 4. They will be used in the course on Determinantal singularities.
- Please do all the exercises in the notes. Whenever you are trying to understand a definition, make up an easy problem and solve it, if there isn't one already. We will have an exercise session next week.
- I encourage you to talk to me at any time during the school if you have any questions.

I Introduction and Some Basic Examples

- To understand a singularity X we want to understand the "nearby" singularities–the singularities that appear in the deformations of X.
- Given a family of sets or maps, when are all the members the same?
- When are some of the members different?
- **Equisingularity** is the study of these questions.
- Advantage: Easier to say when all the members of family are the same, than when two individual sets or two maps are the same.
- Often the change in a single invariant suffices to pick out the members which different than the rest.
- Infinitesimal methods natural and powerful for the study of families.
- Invariants of ICIS have both a topological/ geometric and infinitesimal character. Hypersurface case: $\mu(f)$ is the $T\mathcal{R}_e(f)$ codimension and the rank of the middle homology of the Milnor fiber of f.

Notation

$$\begin{array}{ccccc} X^d(0) \subset & \mathcal{X}^{d+k} \subset & Y \times \mathbb{C}^N \\ \downarrow & \downarrow p_Y & \swarrow \pi_Y \\ 0 \in & Y = \mathbb{C}^k & \end{array}$$

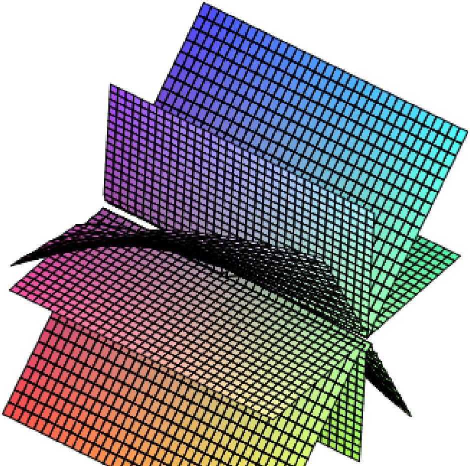
- The parameter space is Y, $X^d(0)$ denotes the fiber of the family over $\{0\}$.
- $\mathcal{X}^{d+k}$ denotes the total space of the family which is contained in $Y \times \mathbb{C}^N$. We always assume $\mathcal{X}^{d+k}$ is equidimensional with equidimensional fibers.
- We usually assume $Y \subset \mathcal{X}^{d+k}$,
- $\mathcal{X} = F^{-1}(0)$, $X(y) = f_y{}^{-1}(0)$, where $f_y(z) = F(y, z)$
- $S(X)$ the singular locus of X.

Definitions and Examples

- The family is *smoothly trivial* if there exists a smooth family of origin preserving bi-holomorphic germs r_y such that $r_y(X(0)) = X(y)$.
- If the map-germs are only homeomorphisms we say the family is C^0 *trivial*.
- **Example:** Let $\mathcal{X}$ be the family of two moving lines in the plane with equation $F(y, z_1, z_2) = z_1(z_2 - yz_1) = 0$.
- y is the parameter, the z_2 axis is fixed, a component of every member of the family while the line $z_2 - yz_1 = 0$ moves with y.
- Our intuition says that all of these sets are the "same".
- In fact, the family of functions $F(y, z_1, z_2) = z_1(z_2 - yz_1)$ are all right equivalent to $f_0(z_1, z_2) = z_1 z_2$, because they are all Morse functions. Hence the family is smoothly trivial.
- Problem: Show that for $y \neq -1$ the family of functions $F(y, z_1, z_2) = z_1 z_2(z_1 - (1 + y)z_2)$ is smoothly trivial: this shows that the family of 3 moving lines in $\mathbb{C}^2$ is smoothly trivial.

4 Moving Lines

- our intuition suggests that the family of n moving distinct lines should be "equisingular". But
- Example: Let $\mathcal{X}$ be the family of four moving lines with equation $F(x, y, z) = z_1 z_2 (z_2 + z_1)(z_2 - (1 + y) z_1) = 0$. The parameter is y, the z_1 and z_2 axes and the line $z_2 + z_1 = 0$ are fixed.



4 Moving Lines continued

- **Problem:** Show that the family of 4 lines is not smoothly trivial.
- Hint: If r_y is a trivialization of the family of sets, $Dr_y(0)$ must carry the tangent lines of $X(0)$ to $X(y)$. If a linear map preserves the lines defined by $z_1 = 0, z_2 = 0, z_2 = -z_1$ then the linear map must be a multiple of the identity. Hence r_y can't map $z_2 = z_1$ to $z_2 = (1 + y)z_1$, $y \neq 0$.

Goal

- The family of four lines is not smoothly trivial, but we still want to use infinitesimal methods as the foundation of our theory of equisingularity.
- The infinitesimal approach using vectorfields, promises to reduce equisingularity problems to algebra, just as Mather's work does for smooth equivalence.
- If not smooth, what kind of vectorfields do we use?

II Rugose vectorfields and Verdier's condition W

- Given a family of hypersurfaces $\mathcal{X}$ over Y^1, defined by $F(y,z)$ consider the vectorfield:
- $V = (\frac{\partial}{\partial y} - \xi)$ defined on $\mathcal{X}_0$,
- $$\xi(y,z) = \frac{\sum_{i=1}^n \frac{\partial F}{\partial y}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)\frac{\partial}{\partial z_i}}{\sum_{i=1}^n \frac{\partial F}{\partial z_i}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)}.$$
- V well-defined and real analytic where $D_z(F) \neq 0$
- $DF(V)(y,z) = F_y(y,z) - \frac{\sum_{i=1}^n \frac{\partial F}{\partial y}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)\frac{\partial F}{\partial z_i}(y,z)}{\sum_{i=1}^n \frac{\partial F}{\partial z_i}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)} = 0$ where defined, which implies V tangent to $\mathcal{X}_0$.
- We want conditions to ensure $\xi(y,0) = 0$, flow of V is at least C^0.

Rugose vectorfields

Theorem

(Verdier) the vectorfield V can be integrated to give a family of homeomorphisms which trivialize $\mathcal{X}$ provided the inequality

$$\|\xi(y,z)\| \leq C\|z\|$$

holds on a neighborhood of the origin in $\mathcal{X}$, for some $C > 0$.

- Verdier called such a vectorfield a *rugose* vectorfield.
- Verdier also defined a stratification condition *condition W*, which ensured, that if it held between all pairs of incident strata, smooth vectorfields on the smallest stratum lifted to rugose vectorfields on larger strata.

Condition W: Distance between linear spaces.

- Suppose A, B are linear subspaces at the origin in $\mathbb{C}^N$
- $$\mathrm{dist}(A,B) = \sup_{\substack{u \in B^{\perp} - \{0\} \\ v \in A - \{0\}}} \frac{\|(u,v)\|}{\|u\|\,\|v\|}.$$
- **Example** we work with linear subspaces of $\mathbb{R}^3$. Let $A = x$-axis, $B \subset \mathbb{R}^3$ a plane with unit normal u_0, then $\mathrm{dist}(A,B) = \|u_0 \cdot (1,0,0)\| = \cos\theta$, where θ is the small angle between u_0 and the x-axis, in the plane they determine. So when the distance is 0, B contains the x-axis.

Definition

Suppose $Y \subset \bar{X}$, where X, Y are strata in a stratification of an analytic space, and $\mathrm{dist}(TY_0, TX_x) \leq C\mathrm{dist}(x, Y)$ for all x close to Y. Then the pair (X, Y) satisfies **Verdier's condition** *W at $0 \in Y$ (Verdier-1976).*

- **Theorem** The set of points of Y where (X_0, Y) satisfy W is Zariski open and dense.

W equisingular families

Definition

A family $\mathcal{X}$ is W-equisingular (or just equisingular) if $\mathcal{X}$ has a stratification in which adjacent pair of strata satisfy condition W, and the parameter space Y is a stratum.

- **Example** The family of n moving distinct lines is W-equisingular because the pair $(\mathcal{X}_0, Y)$ satisfies W, since $\mathcal{X}$ is made up of n smooth surfaces, intersecting along Y, and Y is a submanifold of each smooth surface.
- Since each component of $\mathcal{X}_0$ satisfies W over Y, so does $\mathcal{X}_0$.
- Teissier ('81) showed that Verdier's condition W was equivalent to the Whitney conditions over $\mathbb{C}$. (So, whenever you hear Whitney conditions, you can think W.)

W as an Analytic Inequality

- Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition

Condition W holds for $(\mathcal{X}_0, Y)$ *at* $(0,0)$ *if and only if there exists* U *a neighborhood of* $(0,0)$ *in* $\mathcal{X}$ *and* $C > 0$ *such that*

$$\|\frac{\partial F}{\partial y_l}(y,z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y,z)\|$$

for all $(y,z) \in U$ *and for* $1 \leq l \leq k$.

- Proof: Set $A = Y$, and calculate the distance between Y and a tangent plane to $\mathcal{X}_0$ at (y,z) which is our B.
- Use $\overline{DF(y,z)}/\|DF(y,z)\|$ for $u \in B^{\perp}$, standard basis for the vectors from A.
- distance formula says that condition W holds if and only if

Proof continued

- $$\sup_{1\le l\le k} \frac{\|\frac{\partial F}{\partial y_l}(y,z)\|}{\|DF(y,z)\|} \le C''\mathrm{dist}((y,z),Y) = C' \sup_{1\le i\le n+1} \|z_i\|$$
- This is equivalent to
$$\|\frac{\partial F}{\partial y_l}(y,z)\| \le C \sup_{1\le i\le n+1} \|z_i\| \sup_{1\le j\le n+1} \|\frac{\partial F}{\partial z_j}(y,z)\|$$
- Which gives the result.
- Denote the ideal generated by the partial derivatives of F with respect to the z variables by $J_z(F)$, and the ideal generated by z_j by m_Y. Then $z_i\frac{\partial F}{\partial z_j}$ are a set of generators for $m_Y J_z(F)$.
- The inequality above says that the partial derivatives of F with respect to y_l go to zero as fast as the ideal $m_Y J_z(F)$ does.

III The Theory of Integral Closure of Ideals and Modules

- We want to describe algebraically what it means for a function to go to zero as fast as an ideal does.
- f is integrally dependent on an ideal I if one of the following equivalent conditions obtain:
- (i) There exists a positive integer k and elements a_j in I^j, so that f satisfies the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $\mathcal{O}_{X,0}$.
- (ii) There exists a neighborhood U of 0 in $\mathbb{C}^N$, a positive real number C, representatives of the space germ $X, 0$ the function germ f, and generators $g_1, \ldots, g_m$ of I on U, which we identify with the corresponding germs, so that for all x in U we have: $\|f(x)\| \leq C \max\{\|g_1(x)\|, \ldots, \|g_m(x)\|\}$.
- (iii) For all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$ the pull–back $\phi^* f = f \circ \phi$ is contained in the ideal generated by $\phi^*(I)$ in the local ring of $\mathbb{C}$ at 0. If for all paths $\phi^* f$ is contained in $\phi^*(I)m_1$, then we say f *is strictly dependent on* I and write $f \in I^\dagger$.

Integral closure of ideals continued

- The set of all elements of $\mathcal{O}_{X,x}$ which are integrally dependent on I is the *integral closure of I* and is denoted $\overline{I}$.

Proposition

If I is an ideal in $\mathcal{O}_{X,x}$, then so is $\overline{I}$.

- Proof: We use property iii). Let $\phi : (\mathbb{C}, 0) \to (X, 0)$ be any analytic curve, $g \in \mathcal{O}_{X,x}$, f_1, f_2 in $\overline{I}$.
- Then $(gf_1 + f_2) \circ \phi = (g \circ \phi)(f_1 \circ \phi) + (f_2 \circ \phi) \in \phi^*(I)$, since $\phi^*(I)$ is an ideal in $\mathcal{O}_1$.

Example

Let $A = \mathcal{O}_2$, $I = (x^n, y^n)$. Suppose $f = x^i y^j$, $i + j \geq n$. Consider the monic polynomial $h(T) = T^n - (x^n)^i (y^n)^j$. Since $(x^n)^i (y^n)^j$ is in $(I^i)(I^j) \subset I^{i+j} \subset I^n$, and $h(f) = 0$, then $f \in \overline{I}$, and $\overline{I} \supset m_2^n$.

Hypersurfaces, W and Integral Closure

Proposition

Condition W holds for $(\mathcal{X}_0, Y)$ *at* $(0,0)$ *if and only if* $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ *for* $1 \leq l \leq k$.

- Proof: W holds if and only if

$$\|\frac{\partial F}{\partial y_l}(y,z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y,z)\|$$

- By property 2 this is equivalent to $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$.
- **Problem** Show that the family of hypersurfaces in $\mathbb{C}^3$ defined by $F = x^n + y^n + th, h \in m_2^{n+1}$ is W equisingular.
- What about higher codimension sets?

The Theory of Integral Closure for Modules: Motivation

- Verdier's condition W is based on the distance between the tangent space TX_x to X at smooth points x and the tangent space T to Y.
- Recall

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^{\perp} - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

- If $u \in TX_x^{\perp} - \{0\}$, then the set of points perpendicular to u consists of a hyperplane which contains TX_x. These hyperplanes are called *tangent hyperplanes*; denote a tangent hyperplane to X, x by H_x, and the collection of all tangent hyperplanes to X, x by $C(X)_x$.
- the distance formula becomes:

$$\operatorname{dist}(T, TX_x) = \sup_{H_x \in C(X)_x} \operatorname{dist}(T, H_x)$$

The Jacobian Module and Tangent Hyperplanes

- If $X = F^{-1}(0)$ where $F\colon \mathbb{C}^n \to \mathbb{C}^p$, $F = (f_1, \ldots, f_p)$ then at a smooth point x of X, the entries of $Df_i(x)$ are the coordinates of linear form defining a tangent hyperplane.
- So, the projectivisation of the rowspace of the matrix of partial derivatives of F is $C(X)_p$.
- Since the tangent hyperplanes control the distance between the tangent space of X, p and $TY, 0$, this suggests looking at the module generated by the partial derivatives of F denoted $JM(X)$, just as we looked at $J(F)$ in the hypersurface case.
- What does $\overline{JM(X)}$ mean?

Basic Results from the Theory of Integral Closure for Modules

- Notation: $M \subset N \subset F^p$, F^p a free $\mathcal{O}_{X,x}$ module of rank p, M, N submodules of F.
- If M is generated by g generators $\{m_i\}$, then let $[M]$ be the matrix of generators whose columns are the $\{m_i\}$.

Definition

If $h \in F^p$ then h is integrally dependent on M, if for all curves ϕ, $h \circ \phi \in \phi^*(M)$. The integral closure of M denoted $\overline{M}$ consists of all h integrally dependent on M.

- **Problem** $\overline{M}$ is a module, $\overline{\overline{M}} = \overline{M}$.
- **Example** Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, then $\overline{M} = m_2 \mathcal{O}_2^2$.

Module Analogue of Property 2 for Ideals

Proposition

(Gaffney-1992, Prop 1.11) Suppose $h \in \mathcal{O}^p_{X,x}$ *, M a submodule of* $\mathcal{O}^p_{X,x}$ *of generic rank k on each component of X. Then* $h \in \overline{M}$ *if and only if for each choice of generators* $\{s_i\}$ *of M, there exists a constant* $C > 0$ *and a neighborhood U of x such that for all* $\psi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$,

$$\|\psi(z) \cdot h(z)\| \leq C \sup_i \|\psi(z) \cdot s_i(z)\|$$

for all $z \in U$.

- $(\psi(z) \cdot s_1(z), \dots \psi(z) \cdot s_n(z))$ is a linear combination of the rows of a matrix of generators of M.
- So the property is comparing the size of row vectors of M with the corresponding element of h
- The constant C and the neighborhood U depend on h and M but not on ψ.

Module Analogue of Property 2 for Ideals

Corollary

Suppose $h \in \mathcal{O}^p_{X,x}$, M a submodule of $\mathcal{O}^p_{X,x}$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $T \in \mathbb{C}^p$,

$$\|T \cdot h(z)\| \leq C \sup_i \|T \cdot s_i(z)\|$$

for all $z \in U$.

- Proof: Assume

$$\|\psi(z) \cdot h(z)\| \leq C \sup_i \|\psi(z) \cdot s_i(z)\|$$

for all $z \in U$, then take ψ to be the constant T; conversely, we can replace T by ψ, using the fact that the constant C is independent of the choice of T.

IV Analytic spaces, W and Integral Closure

- Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ an equidimensional family of equidimensional sets, $\mathcal{X}^{k+n} \subset Y^k \times \mathbb{C}^N$, $JM(X) \subset \mathcal{O}^p$.

Theorem

Condition W holds for $(\mathcal{X}_0, Y)$ at $(0,0)$ if and only if $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$.

- Proof: We re-work the form of Verdier's condition W to fit our current framework. If we work at a smooth point x of X, then a conormal vector u of X at x can always be written as $S \cdot DF(x)$, where $S \in \mathbb{C}^p$; S is not unique unless $DF(x)$ has rank p.
- Conversely, any such S gives a conormal vector. It is clear also that W holds if the distance inequality holds for the standard basis for the tangent space T of Y. Then

proof continued I

- $$\mathrm{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^\perp - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

  becomes

- $$\mathrm{dist}(T, TX_x) = \sup_{\substack{S \in \mathbb{C}^p - \{0\} \\ 1 \leq i \leq k, S \cdot DF(x) \neq 0}} \frac{\|S \cdot \frac{\partial f}{\partial y_i}\|}{\|S \cdot DF(x)\|}$$

  because $\|u\| = \|S \cdot DF(x)\|$, and $\|v\| = 1$.

proof continued II

- So Verdier's condition W becomes:
- $$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \le i \le k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \le C\|z\| \, \|S \cdot DF(x)\| \, .$$
- Since the functions are analytic and the inequality holds on a Z-open set of X, we can assume it holds on a neighborhood of the origin.
- consider the integral closure condition, $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \le l \le k$. Using the last corollary, we have $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \le l \le k$ if and only if
- $$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \le i \le k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \le C \sup_{1 \le i \le n} \|z_i S \cdot DF(x)\| \, .$$
- But this is easily seen to be equivalent to the previous inequality which finishes the proof.

$JM(F)$ vs. $JM_z(F)$

- Let $JM_Y(F)$ denote the submodule of $JM(F)$ generated by $\frac{\partial F}{\partial y_l}, 1 \leq l \leq k$, $JM_z(F)$ the partials with respect to z.

Proposition

$JM_Y(F) \subset \overline{m_Y JM(F)}$ if and only if $JM_Y(F) \subset \overline{m_Y JM_z(F)}$.

- Proof: Clear that $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ implies $JM_Y(F) \subset \overline{m_Y JM(F)}$.
- Let ϕ be any curve on $\mathcal{X}, 0$. Then
- $$\begin{aligned} \phi^*(JM_z(F)) \subset \phi^* JM(F) &= \phi^* JM_z(F) + \phi^* JM_Y(F) \\ &= \phi^* JM_z(F) + m_1 \phi^* JM(F). \end{aligned}$$
- Then by Nakayama's lemma, $\phi^*(JM_z(F)) = \phi^* JM(F)$ and $\phi^*(m_Y JM_z(F)) = \phi^*(m_Y JM(F))$.
- Since ϕ is arbitrary, $\overline{m_Y JM_z(F)} = \overline{m_Y JM(F)}$.

Families of ICIS

- For W, we compare the size of row vectors of $DF(x)$, with the part of the row vector coming from $D_Y F(x)$. Since the part coming from $D_Y F(x)$ must go to zero faster than the whole vector, it is sensible that it suffices to compare it with the part coming from $D_z F$. The last proposition justifies this intuition.
- Given a family of ICIS in $Y^k \times \mathbb{C}^n$, $JM_z(F)|\mathcal{X}(y) = JM(X_y^d)$ and this has finite colength in $\mathcal{O}_{X_y}^{n-d}$, so $e(JM(X_y^d, \mathcal{O}_{X_y}^{n-d}, 0)$ multiplicity of $JM(X_y^d)$ in $\mathcal{O}_{X_y}^{n-d}$ is well defined.
- In the next sections we will see how the multiplicity can be used to give necessary and sufficient conditions for W to hold for a family of ICIS.
- Questions?

Lecture 4: Equisingularity and ICIS Part 2

by Terence Gaffney

V Multiplicities of Ideals and Modules

- The multiplicity of an ideal or module or pair of modules is one of the most important invariants we can associate to an m-primary module.
- It is intimately connected with integral closure.
- It has both a length theoretic definition and intersection theoretic definition.
- In the applications it has both an infinitesimal and topological/ geometric interpretation.

Basic Constructions for ideals and modules

- Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators called the Rees algebra of M.
- If $(m_1, \ldots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$.
- $\mathcal{M}_n$ is the terms of $\mathcal{R}(M)$ of degree n.
- $\mathrm{Projan}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal.
- $\mathrm{Projan}(\mathcal{R}(JM(X)))$ is the conormal space of X. It consists of the tangent hyperplanes to X_0 and the closure of this space in $X \times \mathbb{P}^{n-1}$, $X \subset \mathbb{C}^n$.
- Denote the projection to X^d by c, or by c_M where there is ambiguity.

Length Theoretic Definition of Multiplicity

- Denote the length of a module M by $l(M)$.

Theorem/Definition

(Buchsbaum-Rim-1963) Suppose $M \subset F$, M, F both A-modules, F free of rank p, A a Noetherian local ring of dimension d, F/M of finite length, $\mathcal{F} = A[T_1, \ldots, T_p]$, $\mathcal{R}(M) \subset \mathcal{F}$, then $\lambda(n) = l(\mathcal{F}_n/\mathcal{M}_n)$ is eventually a polynomial $P(M, F)$ of degree d+p-1. Writing the leading coefficient of $P(M, F)$ as $e(M)/(d+p-1)!$, then we define $e(M)$ as the multiplicity of M.

- **Example** Claim: Let $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. Then $e(M) = 4$.
  - We have $p = 1$, $F = \mathcal{O}_2$, and we work with $\mathcal{F} = \mathcal{O}_2[T_1]$. (Notice that $\mathrm{Projan}\mathcal{F} = \mathbb{C}^2$.)
  - Now $\mathcal{M}_n = I^n T^n = m_2^{2n} T^n$, so
  -
$$l(\mathcal{F}_n/\mathcal{M}_n) = l(\mathcal{O}_2/m^{2n}) = (2n)(2n+1)/2 = 4n^2/2! + (l.o.t.)$$
  - So $e(M) = 4$.

Geometric meaning of multiplicity

- Suppose $I = (f_1, \ldots, f_d) \subset \mathcal{O}_{X^d}$, then $e(I) =$deg f, where $f = (f_1, \ldots, f_d)\colon X^d \to \mathbb{C}^d$.
- Suppose I has more than d generators; find $J = (f_1, \ldots, f_d) \subset I$ and $\overline{J} = \overline{I}$, then $e(I) =$deg f, where $f = (f_1, \ldots, f_d)\colon X^d \to \mathbb{C}^d$.
- J is called a *reduction* of I.
- Suppose $M \subset \mathcal{O}^p_{X^d}$ has $d + p - 1$ generators. Then $e(M, 0) =$ number of times we count 0 as a point where the rank of $M < p$. If M has more than $d + p - 1$ generators, take a reduction with $d + p - 1$ generators as before.

Example

- **Example** Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. $e(M) = 3$, M has 2+2-1=3 generators.
- Let $[M(t)] = \begin{bmatrix} x-t & y-t & 0 \\ 0 & x & y \end{bmatrix}$
- Then for $t \neq 0$, $[M(t)]$ has rank 1 only at $(0,0), (t,0), (t,t)$.
- So for M we should count $(0,0)$ three times when counting the number of points where the rank of $M < 2$, and $e(M)$ does this.
- For a $p \times (d+p-1)$ matrix, the expected codimension of the set of points where the matrix has less than maximal rank is d.
- This explains why we use $d+p-1$ generators for the reduction.

Reductions of Modules

- Goal: Be able to show $\overline{m_Y JM_z(F)} = \overline{JM_Y(F) + m_Y JM(F)}$
- Given $M \subset F^p$, R a submodule of M with $\overline{M} = \overline{R}$. Then R is called a *reduction* of M.
- If $M \subset N \subset F^p$ or h is a section of N, then h and M generate ideals on $\mathrm{Projan}\mathcal{R}(N)$. Denote them by $\rho(h)$ and $\mathcal{M}$.
- If $h = \sum g_i n_i$, $\{n_i\}$ a set of generators of N then in the chart in which $T_1 \neq 0$, we have:
- $\rho(h) = \sum g_i T_i / T_1$.

Example

If M is the Jacobian module of X and $N = F^p$ then $V(\mathcal{M})$ consists of pairs (x, L) where $x \in X$ and $L \in \mathbb{P}Hom(\mathbb{C}^p, \mathbb{C})$, and $L \circ DF(x) = 0$. If H is the hyperplane which is the kernel of L, then the image of $DF(x)$ lies in H.

- Looking at (M, N) allows us to "strip out" one copy of N from M, as the following example shows.

Reductions II

Example

Let $M = I = (x^2, xy, z) = J(z^2 - x^2y)$ *and* $N = J = (x, z)$*.* M *is the Jacobian ideal of the Whitney umbrella, and* N *defines the singular locus of the umbrella.*

- working on $\mathbb{C}^3$, $\mathcal{R}(N) = \mathcal{O}_3[xS, zS]$ is isomorphic to $R = \mathcal{O}_3[T_1, T_2]/(zT_1 - xT_2)$, by $xS \to T_1$, $zS \to T_2$.
- This shows $B_J(\mathbb{C}^3) = \mathrm{Projan}\mathcal{R}(N)$.
- Since $x^2 = x \cdot x, xy = y \cdot x, z = z$ the map from $\mathcal{R}(I)$ to R has image (xT_1, yT_1, T_2):
- this induces the ideal sheaf $\mathcal{I}$ on $\mathrm{Projan}\mathcal{R}(N)$, which is supported only at the point $(0, [1, 0])$.

Reductions III

Proposition

Suppose $M \subset N \subset \mathcal{O}^p_{X,0}$ *are* $\mathcal{O}^p_X$ *modules with matrix of generators* $[M]$, $[N]$, *and* $[F]$ *is a matrix such that* $[M] = [N][F]$. *Let* $\mathcal{F}$ *be the ideal sheaf induced on* $\mathrm{Projan}(\mathcal{R}(N))$ *by the module* F *with matrix of generators* $[F]$. *Then* $\overline{M} = \overline{N}$ *if and only if* $V(\mathcal{F})$ *is empty.*

- Cf the notes in the section "Blowing up modules and Connection with Ideals II".
- **Problem** Find $[F]$ if $M = (x^2, y^2)$, $N = (x^2, y^2, xy)$, and show $V(F)$ is empty.
- $M \subset N$, $\mathcal{F}$ as above, then the inclusion $i\colon M \to N$ induces a map π_M from $\mathrm{Projan}(\mathcal{R}(N)) \setminus V(\mathcal{F})$ to $\mathrm{Projan}(\mathcal{R}(M))$.
- $\pi_M(x, p) = (x, \mathcal{F}(p))$, where $\mathcal{F}(p)$ is evaluation at p of the set of generators of $\mathcal{F}$ which come from the columns of $[F]$.

Reductions IV

Corollary

Suppose M and N as above, then the following are equivalent:

1. *M is reduction of N.*
2. *$V(\mathcal{F})$ is empty.*
3. *The induced map π_M is a finite map from $\mathrm{Projan}(\mathcal{R}(N))$ to $\mathrm{Projan}(\mathcal{R}(M))$.*

Proposition

Suppose $N \subset F$, F a free $\mathcal{O}_{X,x}$ module, and suppose the fiber of $\mathrm{Projan}\mathcal{R}(N)$ over x has dimension k. Then N has a reduction M, where M is generated by $k+1$ elements.

- **Proof:** $\mathrm{Projan}\mathcal{R}(N) \subset X \times \mathbb{P}^{g-1}$
- Choose plane P in $\mathbb{P}^{g-1}$ of codimension $k+1$ so the intersection of P and the fiber of $\mathrm{Projan}\mathcal{R}(N)$ over x is empty.

Proof of the Proposition continued

- Choose coordinates on $\mathbb{P}^{g-1}$ so P given by $T_1 = \cdots = T_{k+1} = 0$
- Choosing coordinates on $\mathbb{P}^{g-1}$ is equivalent to choosing generators on N.
- Let M be the submodule of N generated by the first $k+1$ generators of N after the new choice of generators. Then the projection onto the first $k+1$ coordinates of $\mathbb{P}^{g-1}$, when restricted to $\mathrm{Projan}\mathcal{R}(N)$ gives a finite map to $\mathrm{Projan}\mathcal{R}(M)$. Hence M is a reduction of N by 3).

Corollary

Suppose $N \subset F$, F a free $\mathcal{O}_{X,x}$ module, X^d equidimensional, N has generic rank e on each component of X, x, then N has a reduction with $d+e-1$ generators.

- **Proof:** Generic rank of N is e, so the generic fiber dimension of $\mathrm{Projan}\mathcal{R}(N)$ is $e-1$, and $\mathrm{d}im\mathrm{Projan}\mathcal{R}(N) = d+e-1$.
- Then $d+e-2$ is the largest the dimension of the fiber of $\mathrm{Projan}\mathcal{R}(N)$ over x can be, so N has a reduction with $(d+e-2)+1$ generators.

Exercises

- **Problem** Let $JM(X,0)_H$ denote the submodule of $JM(X)$ generated by $\{DF(V)\}$, $V \in H$. Show that if H is a hyperplane, then $JM(X,0)_H$ is a reduction of $JM(X,0)$ iff H is not a limiting tangent hyperplane of X at 0. (Hint: Show $V(\mathcal{F})$ is empty.)

Reductions, multiplicity and Cohen-Macauley Rings

Theorem

(Rees) Suppose $M \subset N$ are m primary submodules of F^p, and $\overline{M} = \overline{N}$. Then $e(M) = e(N)$. Suppose further that $\mathcal{O}_{X,x}$ is equidimensional, then $e(M) = e(N)$ implies $\overline{M} = \overline{N}$.

- **Proof:**Kleiman-Thorup-1994.
- **Remark:** If $\mathcal{O}_{X^d,x}$ is Cohen-Macaulay, and $M \subset F^p$ has $d+p-1$ generators, then
- $e(M) =$ colength $M =$ colength $J(M)$, the ideal of maximal minors of M. (Buchsbaum-Rim-1963, 2.4 p.207, 4.3 and 4.5 p.223.)

Proposition

Let $X^1, 0 \subset \mathbb{C}^n, 0$ be an ICIS, defined by $f = (f_1, \ldots, f_{n-1})$, where f_i is homogeneous of degree d_i. Then

$$e(JM(X)) = \left(\sum_{i=1}^{n-1}(d_i - 1)\right)\left(\prod_{i=1}^{n-1} d_i\right).$$

Multiplicity and Lines in Space

- **Proof:** X consists of $(\prod_{i=1}^{n-1} d_i)$ lines, by Bezout's theorem.
- Choose $n-1$ columns of the matrix of partial derivatives, such that the submatrix, $[N]$ gotten has rank $n-1$ on X except at 0. Denote the module the columns generate by N.
- Can assume N is a reduction of $JM(X)$.
- $det[N]$ is homogeneous of degree $(\sum_{i=1}^{n-1}(d_i-1))$, and $e(N)$ =colength of $det[N]$ in $\mathcal{O}_X$, by Buchsbaum-Rim.
- colength of $det[N]$ in $\mathcal{O}_X$ = degree of $det[N]$ as a map from X to $\mathbb{C}$, since $\mathcal{O}_X$ is Cohen-Macauley.
- degree of $det[N]$ on each line in X is the homogeneous degree of $det[N]$
- degree of $det[N]$ on X is the sum of the degrees on each component.
- so

$$e(JM(X)) = e(N) = \left(\sum_{i=1}^{n-1}(d_i-1)\right)\left(\prod_{i=1}^{n-1} d_i\right)$$

e(JM(X)) and the Lê-Greuel Theorem

Proposition

(Module form of the Lê-Greuel formula) Let $X^d, 0$ be an ICIS, $d > 0$, H a hyperplane which is not a limit tangent hyperplane to X at the origin. Then

$$e(JM(X), 0) = \mu(X) + \mu(X \cap H).$$

- Recall, Lê-74 and Greuel-75 proved the following formula:

$$\mu(X) + \mu(X') = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{I},$$

where X is the ICIS defined by $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$; F the map with components $f_1, \ldots, f_k$ and X' the ICIS defined by $F'\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{k+1}, 0)$; F' the map with components $f_1, \ldots, f_{k+1}$, and I is the ideal generated by $f_1, \ldots, f_k$, and the $k+1 \times k+1$-minors $\frac{\partial(f_1,\ldots,f_{k+1})}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}$.

Lê-Greuel II

- **Proof:**
- Let L be the linear form defining H. Let L be f_{k+1} in the formula.
- The right hand side of the formula becomes $\mu(X) + \mu(X \cap H)$.
- $e(JM(X)_H) = e(JM(X))$ since H is a hyperplane which is not a limit tangent hyperplane to X at the origin.
- the ideal of $k+1 \times k+1$ minors of a matrix of generators of $JM(X \cap H)$ is the same as the ideal of $k \times k$ minors of a matrix of generators of $JM(X)_H$.
- This implies that the colength of I in the formula is the colength of $k \times k$ minors of $JM(X)_H$, which by the Buchsbaum-Rim theorem is $e(JM(X)_H)$.

Calculating Milnor Numbers Inductively

Proposition

Suppose I defines an ICIS X of dimension 0*; then* $\mu(X) = e(I, \mathcal{O}_n) - 1$

- **Proof:** I an ICIS implies $I = (f_1, \ldots, f_n)$.
- Then $e(I) = deg(f_1, \ldots, f_n)$ at 0 as a map f from $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$.
- $\# f^{-1}(p) = e(I)$, p not a critical value.
- Fixing one point, as a common point for every 0 sphere, we get a bouquet of $(e(I) - 1)$ 0-spheres. So the Milnor number is $e(I, \mathcal{O}_n) - 1$.

Calculating Milnor Numbers: An Example

Corollary

Let X^1 be a homogeneous ICIS, then

$$\mu(X) = \left(\sum_{i=1}^{n-1}(d_i - 1)\right)\left(\prod_{i=1}^{n-1} d_i\right) - \prod_{i=1}^{n-1} d_i + 1.$$

- **Proof** $e(JM(X), 0) = \mu(X) + \mu(X \cap H)$.
- Solving for $\mu(X)$ we get

$$\mu(X) = e(JM(X), 0) - \mu(X \cap H).$$

- Since X has dimension 1, $\mu(X \cap H) = m(X) - 1$ by the previous proposition. Since X is a union of lines we know $e(JM(X), 0) = \sum_{i=1}^{n-1}(d_i - 1)) \prod_{i=1}^{n-1} d_i$, while $m(X) - 1 = (\prod_{i=1}^{n-1} d_i) - 1$, from which the result follows.

Principle of Specialization of Integral Dependence (PSID)

- First proved by Teissier-'73 for ideals, our proof uses ideas which appear in Teissier-'80.

Theorem

G-Kleiman '99 (Principle of Specialization of Integral Dependence) Assume that X is equidimensional, and that $y \mapsto e(y)$ is constant on Y^k. Let h be a section of a free $\mathcal{O}_X$ module E whose image in $E(y)$ is integrally dependent on the image of $M(y)$ for all y in a dense Zariski open subset of Y. Then h is integrally dependent on M.

Equisingularity of families of ICIS: Sufficiency

Theorem

Let $\mathcal{X}$ be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y),0))$ is independent of y. Then $X-Y$ is smooth, and the pair $(X-Y,Y)$ satisfies W.

- **Proof:** Since $e(y)$ is upper semi-continuous, there can be no points on $X(y)$ except the origin in the co-support of $mJM(X(y))$; hence $JM(X(y))$ has maximal rank except at 0 so $X(y)$ is smooth except at 0.
- This also implies that $JM_z(\mathcal{X})$ has maximal rank off Y, so $X-Y$ is smooth.
- By the genericity of W, we have $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$ on a Z-open subset of Y. So by the PSID, we have that it holds at all points and the family is W equisingular.

Equisingularity of families of ICIS: Necessity

- Given the product $mJM(X)$, there is an expansion formula which relates $e(mJM(X))$ our infinitesimal invariant to the μ_* invariants, which are our topological/geometric invariants.

Theorem

Suppose $X^d, 0$ is an ICIS, H_i a generic plane of codimension i for X^d then

$$e(mJM(X,0)) = \binom{n-1}{d} m(X,0) + \sum_{i=0}^{d-1} \binom{n-1}{i} e(JM(X \cap H_i, 0))$$

$$= \binom{n-1}{d} (\mu_d(X,0)+1) + \sum_{i=0}^{d-1} \binom{n-1}{i} (\mu_i(X,0) + \mu_{i+1}(X,0))$$

- **Proof** (G-'96)

Corollary

Let $\mathcal{X}$ be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y),0))$ is independent of y. Then the μ_* sequence of $X(y)$ is independent of y.

Equisingularity of families of ICIS: Necessity II

Proof: $\mu_*(X(y))$ sequence is upper semi-continuous in y, as is $e(mJM(X(y),0))$; so, all of the terms in the sum must remain constant, if the value of the sum does.

Theorem

(Necessity) Suppose $\mathcal{X}$ is a family of ICIS, and the pair $(\mathcal{X} - Y, Y)$ satisfies W at the origin. Then, the μ_* sequence of $X(y)$ is independent of y, as is $e(m_y JM(X(y)))$.

- **Proof:** Since the families of generic plane sections also satisfy W by Teissier-'81 (See also the notes for a new proof), it follows that these families are topologically trivial,
- Hence the μ_* sequence of $X(y)$ is independent of y. This implies $e(m_y JM(X(y)))$ is independent of y by the expansion formula.